

On Approximating Multi-Criteria TSP*

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We present approximation algorithms for almost all variants of the multi-criteria traveling salesman problem (TSP), whose performances are independent of the number k of criteria and come close to the approximation ratios obtained for TSP with a single objective function.

We present randomized approximation algorithms for multi-criteria maximum traveling salesman problems (Max-TSP). For multi-criteria Max-STSP, where the edge weights have to be symmetric, we devise an algorithm that achieves an approximation ratio of $2/3 - \varepsilon$. For multi-criteria Max-ATSP, where the edge weights may be asymmetric, we present an algorithm with an approximation ratio of $1/2 - \varepsilon$. Our algorithms work for any fixed number k of objectives. This improves over the previously known algorithms, which achieve ratios of $1/k - \varepsilon$ for multi-criteria Max-STSP and $1/(k+1) - \varepsilon$ for multi-criteria Max-ATSP, and is close to the currently best approximation ratios for Max-STSP and Max-ATSP with a single objective ($61/81 \approx 3/4$ and $2/3$, respectively). To get the approximation ratios of $2/3 - \varepsilon$ and $1/2 - \varepsilon$, respectively, we introduce a decomposition technique for cycle covers. These decompositions are optimal in the sense that no decomposition can always yield more than a fraction of $2/3$ and $1/2$, respectively, of the weight of a cycle cover. Furthermore, we present a deterministic algorithm for bi-criteria Max-STSP that achieves an approximation ratio of $61/243 \approx 1/4$.

Finally, we present a randomized approximation algorithm for the asymmetric multi-criteria minimum TSP with triangle inequality (Min-ATSP). This algorithm achieves a ratio of $\log n + \varepsilon$. For this variant of multi-criteria TSP, this is the first approximation algorithm we are aware of. If the distances fulfil the γ -triangle inequality, its ratio is $1/(1 - \gamma) + \varepsilon$.

1 Multi-Criteria Traveling Salesman Problem

1.1 Traveling Salesman Problem

The traveling salesman problem (TSP) is one of the most famous combinatorial optimization problems. Given a graph, the goal is to find a Hamiltonian cycle of maximum or minimum weight (Max-TSP or Min-TSP).

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An instance of Max-TSP is a complete graph $G = (V, E)$ with edge weights $w : E \rightarrow \mathbb{Q}_+$. The goal is to find a Hamiltonian cycle of maximum weight. The weight of a Hamiltonian cycle (or, more general, of any set of edges) is the sum of the weights of its edges. If G is undirected, we have Max-STSP (symmetric TSP). If G is directed, we obtain Max-ATSP (asymmetric TSP).

An instance of Min-TSP is also a complete graph G with edge weights w that fulfil the triangle inequality: $w(u, v) \leq w(u, x) + w(x, v)$ for all $u, v, x \in V$. The goal is to find a Hamiltonian cycle of minimum weight. We have Min-STSP if G is undirected and Min-ATSP if G is directed. In this paper, we only consider the latter. If we restrict the instances to fulfil the γ -triangle inequality ($w(u, v) \leq \gamma \cdot (w(u, x) + w(x, v))$ for all distinct $u, v, x \in V$ and $\gamma \in [\frac{1}{2}, 1)$), then we obtain Min- γ -ATSP.

All variants introduced are NP-hard and APX-hard (Min- γ -ATSP is hard for $\gamma > \frac{1}{2}$). Thus, we have to content ourselves with approximate solutions. The currently best approximation algorithm for Max-STSP achieves an approximation ratio of $61/81$ [10], and the currently best algorithm for Max-ATSP achieves a ratio of $2/3$ [23]. Min-ATSP can be approximated with a factor of $\frac{2}{3} \cdot \log_2 n$, where n is the number of vertices of the instance [14]. Min- γ -ATSP allows for an approximation ratio of $\min\{\frac{\gamma}{1-\gamma}, \frac{1+\gamma}{2-\gamma-\gamma^3}\}$ [7, 9].

Cycle covers are often used for designing approximation algorithms for the TSP [4, 5, 7, 9, 10, 14, 23, 24]. A cycle cover of a graph is a set of vertex-disjoint cycles such that every vertex is part of exactly one cycle. The general idea is to compute an initial cycle cover and then to join the cycles to obtain a Hamiltonian tour. This technique is called *subtour patching* [17]. Hamiltonian cycles are special cases of cycle covers that consist of a single cycle. Thus, the weight of a maximum-weight cycle cover bounds the weight of a maximum-weight Hamiltonian cycle from above, and the weight of a minimum-weight cycle cover is a lower bound for the weight of a minimum-weight Hamiltonian cycle. Moreover, in contrast to Hamiltonian cycles, cycle covers of maximum or minimum weight can be computed efficiently by reduction to matching problems [1].

1.2 Multi-Criteria Optimization

In many optimization problems, there is more than one objective function. This is also the case for the TSP: We might want to minimize travel time, expenses, number of flight changes, etc., while maximizing, e.g., the number of sights along the way. This leads to k -criteria variants of the TSP (k -C-Max-STSP, k -C-Max-ATSP, k -C-Min-STSP, k -C-Min-ATSP for short; if the number of criteria does not matter, we will also speak of MC-Max-STSP etc.).

With respect to a single criterion the term “optimal solution” is well-defined. However, if several criteria are involved, there is no natural notion of a best choice. Instead, we have to be content with trade-off solutions. The goal of multi-criteria optimization is to cope with this problem. To transfer the concept of optimal solutions to multi-criteria optimization problems, the notion of *Pareto curves* (also known as *Pareto sets* or *efficient sets*) was introduced (cf. Ehrgott [12]). A Pareto curve is a set of solutions that can be considered optimal.

We introduce the following terms only for maximization problems. After that, we briefly state the differences for minimization problems.

An instance of k -C-Max-TSP is a complete graph G with edge weights $w_1, \dots, w_k : E \rightarrow \mathbb{Q}_+$. A Hamiltonian cycle H *dominates* another cycle H' if $w_i(H) \geq w_i(H')$ for all $i \in [k] = \{1, \dots, k\}$ and $w_i(H) > w_i(H')$ for at least one i . This means that H is strictly preferable

to H' . A *Pareto curve* of solutions contains all solutions that are not dominated by another solution. For other maximization problems, k -criteria variants are defined analogously.

Unfortunately, Pareto curves cannot be computed efficiently in many cases: First, they are often of exponential size. Second, because of straightforward reductions from knapsack problems, they are NP-hard to compute even for otherwise easy optimization problems. Third, the TSP is NP-hard already with only one objective function, and optimization problems do not become easier with more objectives involved. Therefore, we have to be satisfied with approximate Pareto curves.

For simpler notation, let $w(H) = (w_1(H), \dots, w_k(H))$. Inequalities are meant component-wise. A set \mathcal{P} of Hamiltonian cycles of V is called an α *approximate Pareto curve* for (G, w) if the following holds: For every Hamiltonian cycle H' , there exists a Hamiltonian cycle $H \in \mathcal{P}$ with $w(H) \geq \alpha w(H')$. We have $\alpha \leq 1$, and a 1 approximate Pareto curve is a Pareto curve. (This is not precisely true if there are several solutions whose objective values agree. Furthermore, with this definition, the set of all feasible solutions forms a Pareto curve. If one allows succinct representations of sets, then many NP-hard multi-criteria problems are trivially solvable in polynomial time. However, our algorithms output sets in a “natural” representation, which allows us to remove dominated solutions from the sets easily. Thus, in our case this subtlety is inconsequential, and we will not elaborate on it for the sake of clarity.)

An algorithm is called an α approximation algorithm if, given G and w , it computes an α approximate Pareto curve. It is called a randomized α approximation if its success probability is at least $1/2$. This success probability can be amplified to $1 - 2^{-m}$ by executing the algorithm m times and taking the union of all sets of solutions. (We can also remove solutions from this union that are dominated by other solutions in the union, but this is not required by the definition of an approximate Pareto curve.) Again, the concepts can easily be transferred to other maximization problems.

Papadimitriou and Yannakakis [26] showed that $(1 - \varepsilon)$ approximate Pareto curves of size polynomial in the instance size and $1/\varepsilon$ exist. The technical requirement for the existence is that the objective values of all solutions for an instance X are bounded from above by $2^{p(N)}$ for some polynomial p , where N is the size of X . This is fulfilled in most optimization problems and in particular in our case. However, they only prove the existence, and for many optimization problems it is unclear how to actually find an approximate Pareto curve.

A *fully polynomial time approximation scheme* (FPTAS) for a multi-criteria optimization problem computes $(1 - \varepsilon)$ approximate Pareto curves in time polynomial in the size of the instance and $1/\varepsilon$ for all $\varepsilon > 0$. Papadimitriou and Yannakakis [26] showed that multi-criteria minimum-weight matching admits a *randomized FPTAS*, i. e., the algorithm succeeds in computing a $(1 - \varepsilon)$ approximate Pareto curve with constant probability. This randomized FPTAS yields also a randomized FPTAS for the multi-criteria maximum-weight cycle cover problem [25].

To define Pareto curves and approximate Pareto curves also for minimization problems, in particular for MC-Min-STSP and MC-Min-ATSP, we have to replace all “ \geq ” and “ $>$ ” above by “ \leq ” and “ $<$ ”. Furthermore, α approximate Pareto curves are then defined for $\alpha \geq 1$ and an FPTAS has to achieve an approximation ratio of $1 + \varepsilon$. There also exists a randomized FPTAS for the multi-criteria minimum-weight cycle cover problem.

1.3 Related Work

A considerable amount of research has been done on multi-criteria TSP. Most work is about heuristics for finding approximate solutions without any worst-case guarantee, e.g., simulated annealing [18], tabu search [20], or genetic algorithms [22]. There are also algorithms for finding the exact Pareto curve, although this takes exponential time [8, 28]. Gupta and Warburton [19] used local search and Lagrangean methods to find locally optimal solutions for the so-called Tchebycheff approach, where the aim is to find a solution that minimizes $\max_{i \in [k]} c_i \cdot w_i(H)$. Paquete and Stützle [27] used local search and Fischer and Richter [15] used dynamic programming methods to solve bi-criteria TSP.

Angel et al. [2, 3] considered Min-STSP restricted to edge weights 1 and 2. They analyze a local search heuristic and prove that it achieves an approximation ratio of $3/2$ for $k = 2$ and of $\frac{2k}{k-1}$ for $k \geq 3$.

Ehrigott [11] considered a variant of MC-Min-STSP, where all objectives are encoded into a single objective by using some norm. He proved approximation ratios between $3/2$ and 2 for this problem, where the ratio depends on the norm used.

Manthey and Ram [25] designed a $(2 + \varepsilon)$ approximation algorithm for MC-Min-STSP and an approximation algorithm for MC-Min- γ -ATSP, which achieves a constant ratio but works only for $\gamma < 1/\sqrt{3} \approx 0.58$. They left open the existence of approximation algorithms for MC-Max-STSP, MC-Max-ATSP, and MC-Min-ATSP.

Bläser et al. [6] devised the first randomized approximation algorithms for MC-Max-STSP and MC-Max-ATSP. Their algorithms achieve ratios of $\frac{1}{k} + \varepsilon$ for k -C-Max-STSP and $\frac{1}{k+1} + \varepsilon$ for k -C-Max-ATSP. They argue that with their approach, only approximation ratios of $\frac{1}{k \pm O(1)}$ can be achieved. Nevertheless, they conjectured that approximation ratios of $\Omega(1/\log k)$ are possible.

For an overview of the literature about multi-criteria optimization, including multi-criteria TSP, we refer to Ehrigott and Gandibleux [13].

1.4 New Results

We devise approximation algorithms for MC-Max-STSP, MC-Max-ATSP, and MC-Min-ATSP. The approximation ratios achieved by our algorithms are independent of the number k of criteria, and they come close to the best approximation ratios known for Max-STSP, Max-ATSP, and Min-ATSP with only a single objective function. Our algorithms work for any number k of criteria.

First, we solve the conjecture of Bläser et al. [6] affirmatively. We even prove a stronger result since the performance ratios of our algorithms are independent of k : For MC-Max-STSP, we achieve a ratio of $2/3 - \varepsilon$, while for MC-Max-ATSP, we achieve a ratio of $1/2 - \varepsilon$ (Section 4). Already for $k = 2$, this is an improvement from $\frac{1}{2} - \varepsilon$ to $\frac{2}{3} - \varepsilon$ for 2-C-Max-STSP and from $\frac{1}{3} - \varepsilon$ to $\frac{1}{2} - \varepsilon$ for 2-C-Max-ATSP. The general idea of our algorithm is sketched in Section 2. After that, we introduce a decomposition technique in Section 3 that will lead to our algorithms. The running-time of our algorithms is polynomial in the input size for any fixed $\varepsilon > 0$ and any fixed number k of criteria.

Furthermore, as a first step towards deterministic approximation algorithms for MC-Max-TSP, we devise an approximation algorithm for 2-C-Max-STSP that achieves an approximation ratio of $61/243 > 1/4$. As a side effect, this proves that for 2-C-Max-STSP, there always exists a single Hamiltonian cycle that already is a $1/3$ approximate Pareto curve. For completeness,

we show that this does not hold for k -C-Max-STSP for $k \geq 3$, for MC-Max-ATSP, or for MC-Min-TSP.

Finally, we devise the first approximation algorithm for MC-Min-ATSP (Section 6). In addition, our algorithm improves on the algorithm for MC-Min- γ -ATSP by Manthey and Ram [25] for $\gamma > 0.55$ (see Figure 4 on page 24), and it is the first approximation algorithm for MC-Min- γ -ATSP for $\gamma \in [\frac{1}{\sqrt{3}}, 1)$. The approximation ratio of our algorithm is $\log n + \varepsilon$ for MC-Min-ATSP, where n is the number of vertices. Furthermore, it is a $\frac{1}{1-\gamma} + \varepsilon$ approximation for MC-Min- γ -ATSP for $\gamma \in [\frac{1}{2}, 1)$. Our algorithm is randomized and its running-time is polynomial in the input size and in $1/\varepsilon$ for any fixed number of criteria.

2 Outline and Idea for MC-Max-TSP

For Max-ATSP, we can easily get a $1/2$ approximation: We compute a maximum-weight cycle cover, and remove the lightest edge of each cycle. In this way, we obtain a collection of paths. Then we add edges to connect the paths, which yields a Hamiltonian cycle. For Max-STSP, this approach yields a ratio of $2/3$ since the length of every cycle is at least three.

Unfortunately, this does not generalize to multi-criteria Max-TSP, even though $(1 + \varepsilon)$ approximate Pareto curves of cycle covers can be computed in polynomial time. The reason is that the term “lightest edge” is usually not well defined: An edge that has little weight with respect to one objective might have a huge weight with respect to another objective. Based on this observation, the basic idea behind our algorithms is the following case distinction: First, if every edge of a cycle cover is a *light-weight edge*, i.e., it contributes only little to the overall weight, then removing one edge does not decrease the total weight by too much. We can choose the edges to be removed such that no objective loses too much of its weight.

Second, if there is one edge that is very heavy with respect to one objective (a *heavy-weight edge*), then we take only this edge from the cycle cover. In this way, we have enough weight for one objective, and we proceed recursively on the remaining graph with $k - 1$ objectives.

In Section 3, we deal with the first case. This includes the definition of when we call an edge a light-weight edge. In Section 4, we present our algorithm, which includes the recursion in case of a heavy-weight edge. The approximation ratios that we achieve come close, i.e., up to an arbitrarily small additive $\varepsilon > 0$, to the $1/2$ and $2/3$ mentioned above for mono-criterion Max-ATSP and Max-STSP.

3 Decompositions

3.1 Asymptotically Optimal Decompositions

From any collection P of paths, we obtain a Hamiltonian cycle just by connecting the endpoints of the paths appropriately. Assume that we are given a cycle cover C . If we can find a collection of paths $P \subseteq C$ (by removing one edge of every cycle of C) with $w(P) \geq \alpha \cdot w(C)$ for some $\alpha \in (0, 1]$, then this would yield an approximate solution for Max-TSP.

Let us define decompositions more formally. Let $\alpha \in (0, 1]$, and let C be a cycle cover. Bläser et al. [6] called a collection $P \subseteq C$ of vertex-disjoint paths an α -*decomposition* of C if $w(P) \geq \alpha w(C)$.

Not every cycle cover possesses an α -decomposition for every α . Let $k \geq 1$ be the number of criteria. Bläser et al. defined $\alpha_k^d \in [0, 1]$ to be the maximum number such that the following

holds: every directed cycle cover C with edge weights $w = (w_1, \dots, w_k)$ that satisfies $w(e) \leq \alpha_k^d \cdot w(C)$ for all $e \in C$ possesses an α_k^d -decomposition. The value $\alpha_k^u \in [0, 1]$ is analogously defined for undirected cycle covers.

We have $\alpha_1^d = \frac{1}{2}$ and $\alpha_1^u = \frac{2}{3}$. We also have $\alpha_k^u \geq \alpha_k^d$ and $\alpha_k^u \leq \alpha_{k-1}^u$ as well as $\alpha_k^d \leq \alpha_{k-1}^d$.

Bläser et al. [6] proved $\alpha_k^d \geq \frac{1}{k+1}$ and $\alpha_k^u \geq \frac{1}{k}$. Furthermore, they proved the existence of $\Omega(1/\log k)$ -decompositions, i.e., $\alpha_k^d, \alpha_k^u \in \Omega(1/\log k)$, which led to their conjecture that $\Omega(1/\log k)$ approximation algorithms might exist. However, their approximation algorithms do not make use of the $\Omega(1/\log k)$ decompositions, and they only achieve ratios of $\frac{1}{k} - \varepsilon$ for k -C-Max-STSP and $\frac{1}{k+1} - \varepsilon$ for k -C-Max-ATSP. In fact, they indicate that approximation ratios of $\frac{1}{k+O(1)}$ are the best that can be proved using their approach.

For completeness, we make their decomposition result more precise with the next theorem. In particular, we show that $\alpha_k^d, \alpha_k^u \in \Theta(1/\log k)$, which proves that better approximations require a different decomposition technique. The new decompositions will be introduced in Section 3.2.

Theorem 3.1. *For all $1 \leq k \in \mathbb{N}$, we have*

$$\frac{1}{0.78 \cdot \log_2 k + \frac{3}{2}} \approx \frac{1}{\frac{9}{8} \cdot \ln k + \frac{3}{2}} \leq \alpha_k^u \leq \frac{1}{\lfloor \log_3 k \rfloor + 1} \approx \frac{1}{0.63 \cdot \log_2 k + 1} \text{ and}$$

$$\frac{1}{1.39 \cdot \log_2 k + 4} \approx \frac{1}{2 \cdot \ln k + 4} \leq \alpha_k^d \leq \frac{1}{\lfloor \log_2 k \rfloor + 2}.$$

Proof. We consider the upper bounds first. It suffices to prove $\alpha_k^d \leq \frac{1}{1+\log_2 k}$ for $k = 2^\ell$ and $\alpha_k^u \leq \frac{1}{1+\log_3 k}$ for $k = 3^\ell$ for $\ell \in \mathbb{N}$ since $\alpha_k^u \geq \alpha_{k+1}^u$ and $\alpha_k^d \geq \alpha_{k+1}^d$. Let us first analyze α_k^d for $k = 2^\ell$. For $\ell = 0, 1$, we already know $\alpha_1^d \leq 1/2 = \frac{1}{2+\log_2 1}$ and $\alpha_2^d \leq 1/3 = \frac{1}{2+\log_2 2}$ [6]. So let $\ell \geq 2$.

We now describe a cycle cover C with $w(e) \leq 1$ for all e that does not possess a decomposition P with $w(P) \geq 1$. This proves $\alpha_k^d \leq 1/w_i(C)$ for $i \in [k]$.

Let $\varepsilon > 0$ be a small number that we will specify later on. We hold a “knockout tournament” for the $k = 2^\ell$ objectives. We model a match of $I \subseteq [k]$ versus $J \subseteq [k]$ by a cycle (e, f) with $w_i(e) = 1 - \varepsilon$ for $i \in I$, $w_i(e) = 0$ for $i \notin I$, $w_i(f) = 1 - \varepsilon$ for $i \in J$, and $w_i(f) = 0$ for $i \notin J$. Winning a match thus yields weight $1 - \varepsilon$ as a prize. In the first round, $\{1\}$ plays against $\{2\}$, $\{3\}$ plays $\{4\}$, \dots , and $\{k-1\}$ plays $\{k\}$. In the second round, $\{1, 2\}$ plays $\{3, 4\}$, $\{5, 6\}$ plays $\{7, 8\}$, and so on. In the “semi-final”, which is the $(\ell - 1)$ th round, $\{1, \dots, k/4\}$ plays $\{k/4 + 1, \dots, k/2\}$ while $\{k/2 + 1, \dots, 3k/4\}$ plays $\{3k/4 + 1, \dots, k\}$. The final are three matches were $\{1, \dots, k/2\}$ plays against $\{k/2 + 1, \dots, k\}$. A complete tournament for $\ell = 3$ is shown in Figure 1(a).

We call an objective i a p -underdog if i has lost all its matches up to the p th round.

Lemma 3.2. *For $p \in [\ell]$ and for every $j \in [2^{\ell-p}]$, there is a p -underdog in $\{(j-1) \cdot 2^p + 1, (j-1) \cdot 2^p + 2, \dots, j \cdot 2^p\}$.*

Proof. The proof is by induction on p . For $p = 1$, this is true since $\{(j-1) \cdot 2 + 1\} = \{2j-1\}$ plays $\{2j\}$ and only one of them can win. For $p > 1$, we have one $(p-1)$ -underdog in all $\{(j-1) \cdot 2^{p-1} + 1, \dots, j \cdot 2^{p-1}\}$ for $j \in [2^{\ell-p+1}]$. In every match of the p th round, a team $\{(j-1) \cdot 2^{p-1} + 1, \dots, j \cdot 2^{p-1}\}$ plays a team $\{j \cdot 2^{p-1} + 1, \dots, (j+1) \cdot 2^{p-1}\}$ for an odd j . Either team contains a $(p-1)$ -underdog. One of the $(p-1)$ -underdogs loses again and becomes a p -underdog. \square

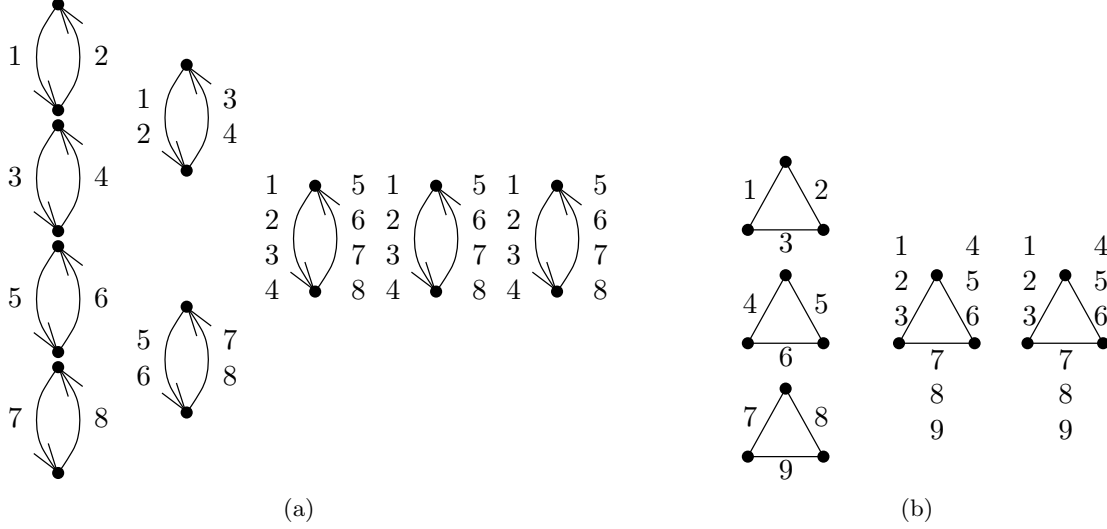


Figure 1: (a) Directed tournament for $8 = 2^3$ objectives. The labels represent the non-zero entries of the corresponding vectors. For instance, label 3,4 represents the vector $(0, 0, 1 - \varepsilon, 1 - \varepsilon, 0, 0, 0, 0)$. (b) Undirected tournament for $9 = 3^2$ objectives.

Thus, before the three final matches, we have two $(\ell - 1)$ -underdogs. In the now following three final matches, these two $(\ell - 1)$ -underdogs play each other. Thus, only one of them can achieve the necessary two victories.

The tournament results stand in one-to-one correspondence to decompositions of the cycle cover C that represents it: We put all “winning” edges into P . We conclude that for every decomposition P of C , there is an i with $w_i(P) \leq 1 - \varepsilon < 1$. Every objective is involved into $\ell + 2$ matches. Thus, $w_i(C) = (\ell + 2) \cdot (1 - \varepsilon)$ and $w_i(e) \leq 1 = \frac{w_i(C)}{(\ell + 2) \cdot (1 - \varepsilon)}$. This implies that C does not possess a $\frac{1}{(\ell + 2) \cdot (1 - \varepsilon)}$ -decomposition. Hence, $\alpha_k^d < \frac{1}{(\ell + 2) \cdot (1 - \varepsilon)}$ for all $\varepsilon > 0$, which yields $\alpha_k^d \leq \frac{1}{\ell + 2} = \frac{1}{2 + \log_2 k}$.

For α_k^u , the proof is similar. The difference is that we now have three teams in every match, and two of the teams win. Let $k = 3^\ell$. In the first round, $\{1\}$ plays $\{2\}$ and $\{3\}$, $\{4\}$ plays $\{5\}$ and $\{6\}$, and so on. In the second round, $\{1, 2, 3\}$ plays $\{4, 5, 6\}$ and $\{7, 8, 9\}$, and so on. We have only two final matches were $\{1, \dots, k/3\}$ plays $\{k/3 + 1, \dots, 2k/3\}$ and $\{2k/3 + 1, \dots, k\}$. Overall, we have $\ell + 1$ matches. See Figure 1(b) for an example. Similar to Lemma 3.2, there exist now three $(\ell - 1)$ -underdogs, and they are in two different teams in the final matches. All three of them must win both finals in order to become a winner, which is impossible. This implies $\alpha_k^u \leq \frac{1}{\ell + 1} = \frac{1}{1 + \log_3 k}$.

The lower bound states that every cycle cover without heavy-weight edges allows for a good decomposition. The proof uses the probabilistic method and Hoeffding’s inequality [21, Theorem 2].

Lemma 3.3 (Hoeffding’s inequality). *Let X_1, \dots, X_n be independent random variables, where X_j assumes values in the interval $[a_j, b_j]$. Let $X = \sum_{j=1}^n X_j$. Then*

$$\mathbb{P}(X < \mathbb{E}(X) - t) \leq \exp \left(- \frac{2t^2}{\sum_{j=1}^n (b_j - a_j)^2} \right).$$

We consider the directed case first. Let $A = 2 \ln k + 4$, and let C be an arbitrary directed cycle cover with $w(e) \leq w(C)/A$ for all $e \in C$. By scaling the edge weights, we make sure that $w_i(C) = A$ and $w_i(e) \in [0, 1]$ for all $i \in [k]$.

Let c_1, \dots, c_m be the cycles of m . Consider any cycle c_j of C . We choose one edge of c_j for removal uniformly at random. By doing this for $j \in [m]$, we obtain a decomposition P of C . Fix any objective i . Let $X_j = \sum_{e \in c_j \cap P} w_i(e)$ be the random variable of the contribution of c_j to the weight $w_i(P)$. Since $w_i(e) \in [0, 1]$ for all $e \in C$, there exists $a_j, b_j \in \mathbb{R}$ such that X_j assumes only values in $[a_j, b_j]$ and $0 \leq b_j - a_j \leq 1$. Let $X = \sum_{j=1}^m X_j = w_i(P)$ be the random variable of the weight of P with respect to objective i . Let $\ell_j \geq 2$ be the length of cycle c_j . A fixed edge of c_j is removed with a probability of $1/\ell_j$. Thus,

$$\mathbb{E}(X) = \sum_{j=1}^m \mathbb{E}(X_j) = \sum_{j=1}^m \left(1 - \frac{1}{\ell_j}\right) \cdot w_i(c_j) \geq \sum_{j=1}^m \frac{1}{2} \cdot w_i(c_j) = \frac{1}{2} \cdot w_i(C) = \frac{A}{2}.$$

If we can show that $\mathbb{P}(X < 1) < 1/k$, then, by a union bound, $\mathbb{P}(\exists i \in [k] : w_i(P) < 1) < 1$, which would imply the existence of a decomposition P with $w(P) \geq 1 = w(C)/A$ and prove the theorem for directed graphs. Since $0 \leq b_j - a_j \leq 1$, we have

$$w_i(C) = A = \sum_{j=1}^m w_i(c_j) \geq \sum_{j=1}^m b_j \geq \sum_{j=1}^m b_j - a_j \geq \sum_{j=1}^m (b_j - a_j)^2.$$

We plug $\mathbb{E}(X) \geq \frac{A}{2}$ and $t = \frac{A}{2} - 1$ and $\sum_{j=1}^m (b_j - a_j)^2 \leq A$ into Hoeffding's inequality and obtain

$$\begin{aligned} \mathbb{P}(X < 1) &\leq \exp\left(-\frac{2(\frac{A}{2} - 1)^2}{A}\right) = \exp\left(-\frac{A}{2} + 2 - \frac{2}{A}\right) \\ &< \exp\left(-\frac{A}{2} + 2\right) = \exp(-\ln k) = \frac{1}{k}. \end{aligned}$$

The proof for undirected graphs goes along the same lines. The differences are that $A = \frac{9}{8} \cdot \ln k + \frac{3}{2}$ and $\mathbb{E}(X) \geq \frac{2A}{3}$. Thus, $t \geq \frac{2A}{3} - 1$, and we obtain

$$\mathbb{P}(X < 1) \leq \exp\left(-\frac{2(\frac{2A}{3} - 1)^2}{A}\right) < \exp\left(-\frac{8A}{9} + \frac{4}{3}\right) = \exp(-\ln k) = \frac{1}{k},$$

which completes the proof of the theorem. \square

3.2 Improved Decompositions

The previous section showed that α -decompositions cannot, in general, yield more than a $\Theta(1/\log k)$ fraction of the weight of a cycle cover. In order to obtain constant approximation ratios, independent of k , we have to generalize the concept of decompositions.

Let C be a cycle cover, and let $w = (w_1, \dots, w_k)$ be edge weights. We say that the pair (C, w) is γ -light for some $\gamma \geq 1$ if $w(e) \leq w(C)/\gamma$ for all $e \in C$.

Theorem 3.4. *Let ε be arbitrary with $0 < \varepsilon < 1/2$, and let $k \geq 2$ be arbitrary. Let C be a cycle cover, and let $w = (w_1, \dots, w_k)$ be edge weights such that (C, w) is $\frac{2 \ln k}{\varepsilon^2}$ -light.*

If C is directed, then there exists a collection $P \subseteq C$ of paths with $w(P) \geq (\frac{1}{2} - \varepsilon) \cdot w(C)$.

If C is undirected, then there exists a collection $P \subseteq C$ of paths with $w(P) \geq (\frac{2}{3} - \varepsilon) \cdot w(C)$.

Proof. The proof uses Hoeffding's inequality. We start by considering the directed case. Let C be a directed cycle cover with edge weights w such that (C, w) is $\frac{2 \ln k}{\varepsilon^2}$ -light. With the same notation as in the proof of the lower bound of Theorem 3.1, we obtain that the probability of the event $w_i(P) < (\frac{1}{2} - \varepsilon) \cdot w_i(C)$ for some fixed i is at most

$$\mathbb{P} \left(w_i(P) < \left(\frac{1}{2} - \varepsilon \right) \cdot w_i(C) \right) \leq \exp \left(- \frac{\varepsilon^2 \cdot \left(\frac{2 \ln k}{\varepsilon^2} \right)^2}{\frac{2 \ln k}{\varepsilon^2}} \right) \leq \frac{1}{k^2} < \frac{1}{k}$$

for $k \geq 2$. The proof for undirected cycle covers is identical and therefore omitted. \square

In fact, the proof above shows that if (C, w) is $(\frac{\ln k}{\varepsilon^2} + \delta)$ -light for any $\delta > 0$, then C admits a $(\frac{1}{2} - \varepsilon)$ - or $(\frac{2}{3} - \varepsilon)$ -decomposition: For any $\delta > 0$, the probability that $w_i(P)$ is too small is strictly less than $1/k$. We have used the (slightly worse) bound of $\frac{2 \ln k}{\varepsilon^2}$ to allow for an efficient randomized algorithm for finding decompositions. For the subsequent sections, we define $\eta_{k,\varepsilon} = \frac{\varepsilon^2}{2 \ln k}$.

To conclude this section, we remark that no better decompositions are possible: First, even if we are content with an $\Omega(1/\log k)$ fraction of the weight, we need an $\Omega(\log k)$ -light cycle cover. This follows from Theorem 3.1.

Second, we cannot hope to get more than one half of the weight in case of directed cycle covers: Consider a cycle cover that consists solely of cycles of length two and where all edges have equal weight. Then we lose one half of the weight no matter how we decompose. Analogously, an undirected cycle cover that consists solely of cycles of length three and where all edges have equal weight shows that we cannot hope to get more than a $2/3$ fraction of the weight. To achieve better ratios than $\frac{2}{3}$ and $\frac{1}{2}$, techniques other than decomposition are needed.

3.3 Finding Decompositions

We know that decompositions exist due to Theorem 3.4. But, in order to use them in approximation algorithms, we have to find them efficiently. In the remainder of this section, we devise two algorithms that do this job, a deterministic one and a faster randomized algorithm.

The randomized algorithm is immediately obtained by exploiting Theorem 3.4: Assume that we have a cycle cover C with edge weights w such that (C, w) is $1/\eta_{k,\varepsilon}$ -light. We randomly select one edge of every cycle of C for removal and put all remaining edges into P . The probability that P is not a $(\frac{1}{2} - \varepsilon)$ - or $(\frac{2}{3} - \varepsilon)$ -decomposition (depending on whether C is directed or undirected) is bounded from above by $1/k \leq 1/2$. Thus, we obtain a decomposition with constant probability. By iterating this until a feasible decomposition has been found, we obtain a Las Vegas algorithm with expected linear running-time.

Theorem 3.5. *For every $\varepsilon > 0$, every $k \geq 2$, and every $1/\eta_{k,\varepsilon}$ -light pair (C, w) , where C is a directed cycle cover, a $(\frac{1}{2} - \varepsilon)$ -decomposition can be computed by a randomized algorithm whose running-time is expected to be linear in the input size.*

If C is an undirected cycle cover, a $(\frac{2}{3} - \varepsilon)$ -decomposition can be computed analogously.

For the deterministic algorithm, we assume again that we have a $1/\eta_{k,\varepsilon}$ -light (C, w) . We scale the weights such that $w_i(C) = 1/\eta_{k,\varepsilon}$ for all i . Then $w(e) \leq 1$ for all $e \in C$. The main idea is to find a kernel, i.e., to reduce an arbitrary instance to a new instance whose size depends only on k and ε .

First, we normalize our cycle cover such that they consist solely of cycles of the shortest possible length. For directed cycle covers C , we can restrict ourselves to cycles of length two:

Any cycle c of length ℓ with edges e_1, \dots, e_ℓ can be replaced by $\lfloor \ell/2 \rfloor$ cycles (e_{2j-1}, e_{2j}) for $j = 1, \dots, \lfloor \ell/2 \rfloor$. If ℓ is odd, then we add an edge $e_{\ell+1}$ with $w(e_{\ell+1}) = 0$ and add the cycle $(e_\ell, e_{\ell+1})$. (Technically, edges consist of vertices, and we cannot simply reconnect them. What we mean is that we create new cycles of length two, whose edges have the same names and the same weights as the original cycles.) We do this for all cycles of length at least three and call the resulting cycle cover C' . Now any decomposition P' of C' yields a decomposition P of the original cycle cover C by removing the newly added edges $e_{\ell+1}$ if they are in P' . Furthermore, $w_i(e) \leq 1$ for the new cycle cover C' . Analogously, undirected cycle covers can be normalized to consist solely of cycles of length three.

Second, assume that we have two cycles c and c' in a normalized cycle cover with $w(c) + w(c') \leq 1$. Then we can combine c and c' to \tilde{c} : Let e_1, e_2 and e'_1, e'_2 be the edges of c and c' , respectively. (For undirected cycles, we proceed analogously.) Then we can replace e_i and e'_i by \tilde{e}_i with $w(\tilde{e}_i) = w(e_i) + w(e'_i)$. The cycle cover plus edge weights thus obtained are still $1/\eta_{k,\varepsilon}$ -light. We continue combining cycles greedily until no more combinations are possible.

The resulting cycle cover contains at most $2k/\eta_{k,\varepsilon}$ cycles. Thus, an optimal decomposition can be found with a running-time that now only depends on k and ε . The normalization can be implemented to run in linear time.

Theorem 3.6. *Fix $\varepsilon > 0$ and $k \geq 2$. Then, for every $1/\eta_{k,\varepsilon}$ -light pair (C, w) , where C is an directed cycle cover, a $(\frac{1}{2} - \varepsilon)$ -decomposition can be computed by a deterministic algorithm whose running-time is polynomial in the input size.*

If C is an undirected cycle cover, a $(\frac{2}{3} - \varepsilon)$ -decomposition can be computed analogously.

We call the procedure described above DECOMPOSE with parameters C , w , and ε : C is a cycle cover (directed or undirected), $w = (w_1, \dots, w_k)$ are k edge weights, and $\varepsilon > 0$. Then $\text{DECOMPOSE}(C, w, \varepsilon)$ returns a $(\frac{1}{2} - \varepsilon)$ - or $(\frac{2}{3} - \varepsilon)$ -decomposition $P \subseteq C$, provided that (C, w) is $\eta_{k,\varepsilon}$ -light.

4 Approximation Algorithms for MC-Max-TSP

In this section, MAXCC-APPROX denotes the randomized FPTAS for cycle covers. More precisely, let G be a graph (directed or undirected), $w = (w_1, \dots, w_k)$ be edge weights, $\varepsilon > 0$ and $p \in (0, 1]$. Then $\text{MAXCC-APPROX}(G, w, k, \varepsilon, p)$ yields a $(1 - \varepsilon)$ -approximate Pareto curve of cycle covers of G with weights w with a success probability of at least $1 - p$.

4.1 Multi-Criteria Max-ATSP

Our goal is now either to use decomposition or to reduce the k -criteria instance to a $(k - 1)$ -criteria instance. To this aim, we put the cart before the horse: Instead of computing Hamiltonian cycles, we assume that they are given. Then we show how to force an algorithm to find approximations to them. To obtain a $1/2 - \varepsilon$ approximate Pareto curve, we have to make sure that for every Hamiltonian cycle \tilde{H} , we have a Hamiltonian cycle H in our set with $w(H) \geq (\frac{1}{2} - \varepsilon) \cdot w(\tilde{H})$.

Fix ε with $0 < \varepsilon < \frac{1}{2 \ln k}$, let \tilde{H} be any Hamiltonian cycle, and let $\beta_i = \max\{w_i(e) \mid e \in \tilde{H}\}$ be the weight of the heaviest edge with respect to the i th objective. Let $\beta = \beta(\tilde{H}) = (\beta_1, \dots, \beta_k)$. We will distinguish two cases.

In the first case, we assume that $\beta \leq (\eta_{k,\varepsilon} - \varepsilon^3) \cdot w(\tilde{H})$, i.e., \tilde{H} does not contain any heavy-weight edges. (Note that $\eta_{k,\varepsilon} - \varepsilon^3 > 0$ since $\eta_{k,\varepsilon} = \frac{\varepsilon^2}{2 \ln k}$ and $\varepsilon < \frac{1}{2 \ln k}$.) We modify our edge weights w to w^β as follows:

$$w^\beta(e) = \begin{cases} w(e) & \text{if } w(e) \leq \beta \text{ and} \\ 0 & \text{if } w_i(e) > \beta_i \text{ for some } i. \end{cases}$$

This means that we set all edge weights exceeding β to 0. Since \tilde{H} does not contain any edges whose weight has been set to 0, we have $w(\tilde{H}) = w^\beta(\tilde{H})$. Furthermore, for all subsets C of edges, we have $w^\beta(C) \leq w(C)$. The advantage of w^β is that, if we compute a $(1 - \varepsilon)$ approximate Pareto curve \mathcal{C}^β of cycle covers with edge weights w^β , we obtain a cycle cover to which we can apply decomposition to obtain a collection P of paths. This set P yields then a tour H that approximates \tilde{H} . This is stated in the following lemma.

Lemma 4.1. *Let $\varepsilon > 0$ be arbitrary. Let \tilde{H} be a directed Hamiltonian cycle with $w(e) \leq (\eta_{k,\varepsilon} - \varepsilon^3) \cdot w(\tilde{H})$ for all $e \in \tilde{H}$. Let $\beta = \beta(\tilde{H})$, and let \mathcal{C}^β be a $(1 - \varepsilon)$ approximate Pareto curve of cycle covers with respect to w^β .*

Then \mathcal{C}^β contains a cycle cover C with $w^\beta(C) \geq (1 - \varepsilon) \cdot w(\tilde{H})$ and $w^\beta(e) \leq \eta_{k,\varepsilon} \cdot w^\beta(C)$ for all $e \in C$. This cycle cover C yields a decomposition $P \subseteq C$ with $w(P) \geq (\frac{1}{2} - 2\varepsilon) \cdot w(\tilde{H})$.

Proof. Since the Hamiltonian cycle \tilde{H} is in particular a cycle cover, the set \mathcal{C}^β contains a cycle cover C with $w^\beta(C) \geq (1 - \varepsilon) \cdot w^\beta(\tilde{H}) = (1 - \varepsilon) \cdot w(\tilde{H})$. For every edge $e \in C$ and every i , we have $w_i^\beta(e) \leq (\eta_{k,\varepsilon} - \varepsilon^3) \cdot w_i^\beta(\tilde{H}) \leq \frac{\eta_{k,\varepsilon} - \varepsilon^3}{1 - \varepsilon} \cdot w_i^\beta(C) \leq \eta_{k,\varepsilon} \cdot w_i(C)$. The last inequality follows from $\eta_{k,\varepsilon} - \varepsilon^3 \leq \eta_{k,\varepsilon} \cdot (1 - \varepsilon)$, which is equivalent to $\varepsilon^3 \geq \eta_{k,\varepsilon} \cdot \varepsilon$. Plugging in $\eta_{k,\varepsilon} = \frac{\varepsilon^2}{\ln k}$ yields $\varepsilon^3 \geq \frac{\varepsilon^3}{2 \ln k}$, which is valid and proves the preceding inequalities.

The cycle cover C can be decomposed into a collection $P \subseteq C$ of paths with $w(P) \geq w^\beta(P) \geq (\frac{1}{2} - \varepsilon) \cdot w^\beta(C)$. Thus, $w(P) \geq (\frac{1}{2} - \varepsilon) \cdot (1 - \varepsilon) \cdot w(\tilde{H}) \geq (\frac{1}{2} - 2\varepsilon) \cdot w(\tilde{H})$. \square

In the second case, we assume that there exists an edge $e = (u, v) \in \tilde{H}$ and an $i \in [k]$ with $w_i(e) > (\eta_{k,\varepsilon} - \varepsilon^3) \cdot w(\tilde{H})$. We put this edge into a set K of edges that we want to have in our cycle cover no matter what. Then we contract the edge e by removing all outgoing edges of u and all incoming edges of v and identifying u and v . (The resulting graph is still complete.) In this way, we obtain a slightly smaller Hamiltonian cycle $\tilde{H}' = \tilde{H} \setminus \{e\}$. Again, there might be an edge $e' \in \tilde{H}'$ and an $i' \in [k]$ with $w_{i'}(e') > (\eta_{k,\varepsilon} - \varepsilon^3) \cdot w_{i'}(\tilde{H}')$. (Since $w(\tilde{H}') \leq w(\tilde{H})$, there can be edges that are heavy with respect to \tilde{H}' that have not been heavy with respect to \tilde{H} .) We put e' into K , contract e' and recurse. How long can this go on? There are two cases that can make an end: First, we might have obtained a Hamiltonian cycle H' that does not have any more heavy-weight edges, i.e., $w(e) \leq (\eta_{k,\varepsilon} - \varepsilon^3) \cdot w(H')$ for all $e \in H'$. In this case, we can apply Lemma 4.1 with decomposition. Second, we might have an $i \in [k]$ with $w_i(K) \geq (\frac{1}{2} - \varepsilon) \cdot w_i(\tilde{H})$, where \tilde{H} is our original Hamiltonian tour. Then we have collected enough weight with respect to the i th objective, and we can continue with only $k - 1$ objectives. The next lemma gives an upper bound on the number of edges that have to be put into K .

Lemma 4.2. *After at most*

$$f(k, \varepsilon) = k \cdot \left\lceil \frac{\log(\frac{1}{2} + \varepsilon)}{\log(1 - \eta_{k,\varepsilon} + \varepsilon^3)} \right\rceil$$

iterations, the procedure described above halts.

Proof. We prove that after at most $f(k, \varepsilon)$ iterations, there exists an i with $w_i(K) \geq (\frac{1}{2} - \varepsilon) \cdot w_i(\tilde{H})$. Let $\tilde{H} = H_0$, and H_ℓ obtained from $H_{\ell-1}$ by contracting one edge as described. Then, for every ℓ , there exists an i such that $w_i(H_\ell) \leq (1 - \eta_{k,\varepsilon} + \varepsilon^3) \cdot w_i(H_{\ell-1})$. Thus, for $\ell = km$, there exists an i with $w_i(H_\ell) \leq (1 - \eta_{k,\varepsilon} + \varepsilon^3)^m \cdot w_i(\tilde{H})$. If K contains km edges, then there exists an i with $w_i(K) \geq w_i(\tilde{H}) \cdot (1 - (1 - \eta_{k,\varepsilon} + \varepsilon^3)^m)$. Now we have

$$\begin{aligned} 1 - (1 - \eta_{k,\varepsilon} + \varepsilon^3)^m \geq \frac{1}{2} - \varepsilon &\Leftrightarrow (1 - \eta_{k,\varepsilon} + \varepsilon^3)^m \leq \frac{1}{2} + \varepsilon \\ &\Leftrightarrow m \cdot \log(1 - \eta_{k,\varepsilon} + \varepsilon^3) \leq \log\left(\frac{1}{2} + \varepsilon\right) \\ &\Leftrightarrow m \geq \frac{\log\left(\frac{1}{2} + \varepsilon\right)}{\log(1 - \eta_{k,\varepsilon} + \varepsilon^3)}. \end{aligned}$$

To complete the proof, we observe that both $\log(\frac{1}{2} + \varepsilon)$ and $\log(1 - \eta_{k,\varepsilon} + \varepsilon^3)$ are negative, the latter since $\eta_{k,\varepsilon} > \varepsilon^3$, which follows from $\eta_{k,\varepsilon} = \frac{\varepsilon^2}{2 \ln k} > \varepsilon^3$ and the choice of $\varepsilon < \frac{1}{2 \ln k}$. \square

To obtain an algorithm, we have to find β and K . So far, we have assumed that we already know the Hamiltonian cycles that we aim for. But there is only a polynomial number of possibilities for β and K : For all β , we can assume that for all i there is an edge with $w_i(e) = \beta_i$. Thus, for every i there are at most $O(n^2)$ choices for β_i , hence at most $O(n^{2k})$ in total. The cardinality of K is bounded in terms of $f(k, \varepsilon)$ as shown in the lemma above. For fixed k and ε , there is only a polynomial number of subsets of cardinality at most $f(k, \varepsilon)$. In fact, we can restrict ourselves to the subsets K that are path covers: A path cover is a subset K of edges such that K does not contain cycles and both the indegree and outdegree of every vertex is at most one.

Overall, we obtain MAXATSP-APPROX (Algorithm 1) and the following theorem.

Theorem 4.3. *For every $k \geq 1$, $\varepsilon > 0$, MAXATSP-APPROX is a randomized $\frac{1}{2} - \varepsilon$ approximation for k -criteria Max-ATSP whose running-time for a success probability of at least $1 - p$ is polynomial in the input size and $\log(1/p)$.*

Proof. We have to estimate three things: approximation ratio, running-time, and success probability. All proofs are by induction on k . For $k = 1$, the theorem holds since there is a deterministic, polynomial-time $2/3$ approximation for mono-criterion Max-ATSP. In the following, we assume that the theorem is correct for $k - 1$, all fixed $\varepsilon > 0$, and all $p > 0$, and we prove it for k , fixed $\varepsilon > 0$, and all $p > 0$.

Let us start by estimating the approximation ratio. For this purpose, we assume that all randomized computations are successful. Let \tilde{H} be an arbitrary Hamiltonian cycle. For a subset $K \subseteq \tilde{H}$, let \tilde{H}_K be \tilde{H} with all edges in K being contracted. Then, by Lemma 4.2, there exists a (possibly empty) set $K \subseteq \tilde{H}$ of edges of cardinality at most $f(k, \varepsilon/2)$ with one of the two following properties:

1. There exists an i with $w_i(K) \geq (\frac{1}{2} - \frac{\varepsilon}{2}) \cdot w_i(\tilde{H})$.
2. For all $e \in \tilde{H}_K$, we have $w(e) \leq (\eta_{k,\varepsilon/2} - (\frac{\varepsilon}{2})^3) \cdot w(\tilde{H}_K)$.

In the first case, there exists an $H' \in \mathcal{P}_{\text{TSP}}^{K,i}$ with $w_j(H') \geq (\frac{1}{2} - \frac{\varepsilon}{2}) \cdot w_j(\tilde{H}_K)$ for all $j \in [k] \setminus \{i\}$. H' combined with K yields a Hamiltonian cycle H that satisfies $w(H) \geq (\frac{1}{2} - \frac{\varepsilon}{2}) \cdot w(\tilde{H})$: First, we have

$$w_i(H) \geq w_i(K) \geq \left(\frac{1}{2} - \frac{\varepsilon}{2}\right) \cdot w_i(\tilde{H}).$$

```

 $\mathcal{P}_{\text{TSP}} \leftarrow \text{MAXATSP-APPROX}(G, w, k, \varepsilon, p)$ 
input: directed complete graph  $G = (V, E)$ ,  $k \geq 1$ , edge weights  $w : E \rightarrow \mathbb{N}^k$ ,  $\varepsilon > 0$ 
output:  $(\frac{1}{2} - \varepsilon)$  approximate Pareto curve  $\mathcal{P}_{\text{TSP}}$  for  $k$ -C-Max-ATSP with a success probability
of at least  $1 - p$ 
1: if  $k = 1$  then
2:   compute a  $2/3$  approximation  $\mathcal{P}_{\text{TSP}}$ 
3: else
4:   for all subsets  $K \subseteq E$  with  $|K| \leq f(k, \varepsilon/2)$  such that  $K$  is a path cover do
5:     contract all edges of  $K$  to obtain  $G_K$ 
6:     for all bounds  $\beta$  of  $(G_K, w)$  do
7:        $\mathcal{C}_{K,\beta} \leftarrow \text{MAXCC-APPROX}(G_K, w^\beta, k, \frac{\varepsilon}{2}, \frac{p}{2n^{2k+2f(k, \varepsilon/2)}})$ 
8:       for all  $C \in \mathcal{C}_{K,\beta}$  with  $w^\beta(e) \leq \eta_{k, \varepsilon/2} \cdot w^\beta(C)$  for all  $e \in C$  do
9:          $P \leftarrow \text{DECOMPOSE}(C, w^\beta, \varepsilon/2)$ 
10:        add edges to  $K \cup P$  to obtain a Hamiltonian cycle  $H$ 
11:        add  $H$  to  $\mathcal{P}_{\text{TSP}}$ 
12:      for all  $i \leftarrow 1$  to  $k$  do
13:        remove the  $i$ th objective from  $w$  to obtain  $w'$ 
14:         $\mathcal{P}_{\text{TSP}}^{K,i} \leftarrow \text{MAXATSP-APPROX}(G_K, w', k-1, \frac{\varepsilon}{2}, \frac{p}{2n^{2k+2f(k, \varepsilon/2)}})$ 
15:        for all  $H' \in \mathcal{P}_{\text{TSP}}^{K,i}$  do
16:           $H \leftarrow K \cup H'$ 
17:          add  $H$  to  $\mathcal{P}_{\text{TSP}}$ 

```

Algorithm 1: Approximation algorithm for MC-Max-ATSP.

Second, for $j \neq i$, we have

$$w_j(H) = w_j(K) + w_j(H') \geq w_j(K) + \left(\frac{1}{2} - \frac{\varepsilon}{2}\right) \cdot w_j(\tilde{H}_K) = \left(\frac{1}{2} - \frac{\varepsilon}{2}\right) \cdot w_j(\tilde{H}).$$

In the second case, let $\beta_i = \max\{w_i(e) \mid e \in \tilde{H}_K\} \leq (\eta_{k, \varepsilon/2} - (\frac{\varepsilon}{2})^3) \cdot w_i(\tilde{H}_K)$. Then, according to Lemma 4.1, $\mathcal{C}_{K,\beta}$ contains a cycle cover C with $w(C) \geq (1 - \frac{\varepsilon}{2}) \cdot w(\tilde{H}_K)$ and $w^\beta(e) \leq \eta_{k, \varepsilon/2} \cdot w(\tilde{H}_K)$. Thus, C can be decomposed into a collection P of paths with $w(P) \geq (\frac{1}{2} - \frac{\varepsilon}{2}) \cdot w(C)$ according to Lemma 4.1. Together with K , this yields a Hamiltonian cycle H with

$$\begin{aligned} w(H) &\geq w(P) + w(K) \geq \left(\frac{1}{2} - \frac{\varepsilon}{2}\right) \cdot w(C) + w(K) \geq \left(\frac{1}{2} - \frac{\varepsilon}{2}\right) \cdot \left(1 - \frac{\varepsilon}{2}\right) \cdot w(\tilde{H}_K) + w(K) \\ &= \left(\frac{1}{2} - \frac{3\varepsilon}{4} + \frac{\varepsilon^2}{4}\right) \cdot w(\tilde{H}_K) + w(K) \geq \left(\frac{1}{2} - \varepsilon\right) \cdot w(\tilde{H}). \end{aligned}$$

Now we analyze the success probability and the running-time simultaneously. By the induction hypothesis, the running-time of MAXATSP-APPROX is polynomial in the input size and in $\log(1/p)$ for a success probability of at least $1 - p$ for $k - 1$ objectives.

Lines 4 and 6 yield at most $n^{2k+2f(k, \varepsilon/2)}$ iterations. Then we compute that many Pareto curves of cycle covers in line 7, each of polynomial cardinality. Some of these cycle covers are then decomposed to yield Hamiltonian cycles for \mathcal{P}_{TSP} . This can be implemented to run in polynomial time.

Lines 4 plus 12 yield $k \cdot n^{2f(k, \varepsilon/2)} \leq n^{2k+2f(k, \varepsilon/2)}$ iterations. Each iteration requires a call of MAXATSP-APPROX with $k - 1$ criteria, which is also polynomial.

We have at most $2n^{2k+2f(k, \varepsilon/2)}$ calls of a randomized procedure, each called with an error bound of $\frac{p}{2n^{2k+2f(k, \varepsilon/2)}}$. By a union bound, this gives an overall error probability of at most p . \square

4.2 Multi-Criteria Max-STSP

The $(\frac{1}{2} - \varepsilon)$ approximation for MC-Max-ATSP works of course also for MC-Max-STSP since the latter is a special case of the former. Our goal, however, is a ratio of $(\frac{2}{3} - \varepsilon)$. As a first attempt, one might just replace the $(\frac{1}{2} - \varepsilon)$ -decompositions by $(\frac{2}{3} - \varepsilon)$ -decompositions. Unfortunately, this is not sufficient since contracting the heavy-weight edges in undirected graphs is not as easy as it is for directed graphs: First, both statements “remove all incoming” and “remove all outgoing” edges are not well-defined in an undirected graph. Second, if we just consider all edges of one vertex as the incoming edges and all edges of the other vertex as the outgoing edges, we obtain a directed graph, which allows only for a ratio of $\frac{1}{2} - \varepsilon$.

To circumvent these problems, we do not contract edges $e = \{u, v\}$. Instead, we set the weight of all edges incident to u or v to 0. This allows us to add the edge e to any Hamiltonian cycle H' without decreasing the weight: We remove all edges incident to u or v from H' , and then we add e . The result is a collection of paths. Then we add edges to connect these paths to a Hamiltonian cycle. The only edges that we have removed are edges incident to u or v , which have weight 0 anyway.

However, by setting the weight of edges adjacent to u or v to 0, we might destroy a lot of weight with respect to some objective. To circumvent this problem as well, we have to consider larger neighborhoods of the edges in K . In this way, we can add our heavy-weight edge (plus some more edges of its neighborhood) to the Hamiltonian cycle without losing too much weight from removing other edges. Lemma 4.4 justifies this.

Before going into the details, let us fix some notation. Let \tilde{H} be an arbitrary Hamiltonian cycle. Let e_0, e_1, \dots, e_{n-1} be the edges of \tilde{H} in the order in which they appear in \tilde{H} (e_0 is chosen arbitrarily). Let $e_j = \{v_j, v_{j+1}\}$, where arithmetic of the indices here and in the following is modulo n .

Lemma 4.4. *Let \tilde{H} be a Hamiltonian cycle as described above, let $w = (w_1, \dots, w_k)$ be edge weights, and let e_1, \dots, e_ℓ be any ℓ distinct edges of \tilde{H} . Then there exists a $j \in [\ell]$ such that*

$$w(e_j) \leq \frac{k}{\ell} \cdot w(\tilde{H}).$$

Proof. Suppose otherwise and assume without loss of generality that $w_i(\tilde{H}) > 0$ for all i . We scale the weights such that $w_i(\tilde{H}) = 1$ for all i . Then for all j there is an i_j with $w_{i_j}(e_j) > \frac{k}{\ell} \cdot w_{i_j}(\tilde{H}) = \frac{k}{\ell}$. Thus, $\sum_{j=1}^{\ell} \sum_{i=1}^k w_i(e_j) > \sum_{j=1}^{\ell} \frac{k}{\ell} \cdot w_{i_j}(\tilde{H}) = k$. But, since all edges are distinct, we also have $\sum_{j=1}^{\ell} \sum_{i=1}^k w_i(e_j) \leq \sum_{i=1}^k w_i(\tilde{H}) = k$, which is a contradiction. \square

Our aim is now again to force the algorithm to find a Hamiltonian cycle H with $w(H) \geq (\frac{2}{3} - \varepsilon) \cdot w(\tilde{H})$. Like for the asymmetric TSP, we distinguish two cases. The first case is that \tilde{H} consists solely of light-weight edges, i.e., $w(e) \leq (\eta_{k, \varepsilon} - \varepsilon^3) \cdot w(\tilde{H})$ for all $e \in \tilde{H}$. We define β and w^β in the same way as in the previous section. The undirected variant of Lemma 4.1 is similar to its directed counterpart.

Lemma 4.5. *Let $\varepsilon > 0$ be arbitrary. Let \tilde{H} be an undirected Hamiltonian cycle with $w(e) \leq (\eta_{k,\varepsilon} - \varepsilon^3) \cdot w(\tilde{H})$ for all $e \in \tilde{H}$. Let $\beta = \beta(\tilde{H})$, and let \mathcal{C}^β be a $(1 - \varepsilon)$ approximate Pareto curve of cycle covers with respect to w^β .*

Then \mathcal{C}^β contains a cycle cover C with $w^\beta(C) \geq (1 - \varepsilon) \cdot w(\tilde{H})$ and $w^\beta(e) \leq \eta_{k,\varepsilon} \cdot w^\beta(C)$ for all $e \in C$. This cycle cover C yields a decomposition $P \subseteq C$ with $w(P) \geq (\frac{2}{3} - 2\varepsilon) \cdot w(\tilde{H})$.

Proof. The proof goes along the same lines as the proof of Lemma 4.1. We just have to replace $1/2$ by $2/3$. The details are therefore omitted. \square

The second case is that \tilde{H} contains a heavy-weight edge. Without loss of generality, let this edge be e_0 . Then we put e_0 into our set K . However, we cannot contract edges since we have an undirected graph. Instead, we set the weight of all edges incident to v_0 and v_1 to 0. But in this way, we lose the weight of e_1 and $e_{-1} = e_{n-1}$. In order to maintain the approximation ratio, we have to avoid that we lose too much weight. Therefore, we consider paths that include e_0 . If we set the weight of all edges incident to $v_j, \dots, v_{j'}$ with $j \leq 0 < j'$ to 0, we lose only the weight of e_{j-1} and $e_{j'}$. To keep track of things, we also put $e_j, \dots, e_{j'-1}$ into K . Furthermore, we put the two edges $e_{j-1}, e_{j'}$, whose weight might get lost, into T . By Lemma 4.4, we can make sure that both e_{j-1} and $e_{j'}$ are not too heavy. Finally, we put $v_j, \dots, v_{j'}$ into the set L , which is the set of vertices whose edge weights have been set to 0. We denote the corresponding edge weights by w^L , i.e., $w^L(e) = 0$ if $e \cap L \neq \emptyset$.

Given any Hamiltonian cycle H' , we can now remove all edges of weight 0, add the edges $e_j, \dots, e_{j'-1}$, and connect the collection of paths to obtain a new Hamiltonian cycle H with $w^L(H) = w^L(H')$. The only edges that we cannot force to be in H are e_{j-1} and $e_{j'}$. In order to maintain a good approximation ratio we have to make sure that both are light with respect to all objectives. This is where Lemma 4.4 comes into play: We choose j and j' such that $w(e_{j-1}) + w(e_{j'}) \leq \nu \cdot w(\tilde{H})$ for some small $\nu > 0$ that will depend on ε and k and that we will specify later on.

As we did for directed graphs, we recurse until either $w_i(K) \geq (\frac{2}{3} - \varepsilon) \cdot w_i(\tilde{H})$ for some i or we have $w^L(e) \leq (\eta_{k,\varepsilon} - \varepsilon^3) \cdot w^L(\tilde{H})$ so that we can apply Lemma 4.5 to w^L and \tilde{H} . The following lemma is the counterpart for undirected graphs of Lemma 4.2.

Lemma 4.6. *Let*

$$g(k, \varepsilon) = k \cdot \left\lceil \frac{\log(\frac{1}{6} + \varepsilon)}{\log(1 - \eta_{k,\varepsilon} + \varepsilon^3)} \right\rceil$$

and $\nu \in (0, \frac{1}{6 \cdot g(k, \varepsilon)}]$. Then the procedure described above halts after at most $g(k, \varepsilon)$ iterations.

Proof. Let $w^{(0)} = w$, and let $w^{(z)}$ be the edge weights after the z th iterations. Similar to the proof of Lemma 4.2, we show that after at most $z \leq g(k, \varepsilon)$ iterations, there exists an i with $w_i^{(z)}(K) \geq (\frac{2}{3} - \varepsilon) \cdot w_i^{(z)}(\tilde{H})$.

For every z , there exists an i with $w_i^{(z)}(\tilde{H}) \leq (1 - \eta_{k,\varepsilon} + \varepsilon^3) \cdot w_i^{(z-1)}(\tilde{H})$. Thus, for $z = km$, there is an i with $w_i^{(z)}(\tilde{H}) \leq (1 - \eta_{k,\varepsilon} + \varepsilon^3)^m \cdot w_i(\tilde{H})$.

Unfortunately and different to the proof of Lemma 4.2, we do not have $w(K) = w(\tilde{H}) - w^{(z)}(\tilde{H})$ since some edges ended up in T . Instead, we only have $w(K) = w(\tilde{H}) - w^{(z)}(\tilde{H}) - w(T)$. We use the inequality $w(T) \leq \nu z w(\tilde{H})$ to bound the weight of T . Thus,

$$w(K) \geq (1 - (1 - \eta_{k,\varepsilon} + \varepsilon^3)^m - \nu km) \cdot w(\tilde{H}).$$

```

 $\mathcal{P}_{\text{TSP}} \leftarrow \text{MAXSTSP-APPROX}(G, w, k, \varepsilon, p)$ 
input: undirected complete graph  $G = (V, E)$ ,  $k \geq 1$ , edge weights  $w : E \rightarrow \mathbb{N}^k$ ,  $\varepsilon > 0$ 
output:  $(\frac{2}{3} - \varepsilon)$  approximate Pareto curve  $\mathcal{P}_{\text{TSP}}$  for  $k$ -C-Max-ATSP with a success probability
of at least  $1 - p$ 
1: if  $k = 1$  then
2:   compute a 61/81 approximation  $\mathcal{P}_{\text{TSP}}$ 
3: else
4:   for all subsets  $K \subseteq E$  with  $|K| \leq h(k, \varepsilon/3)$  such that  $K$  is a path cover do
5:     let  $L$  be the set of vertices incident to  $K$ 
6:     obtain  $w^L$  from  $w$  by setting the weight of all edges incident to  $L$  to 0
7:     for all bounds  $\beta$  of  $(G, w^L)$  do
8:        $\mathcal{C}_{L,\beta} \leftarrow \text{MAXCC-APPROX}(G, w^{L\beta}, k, \frac{\varepsilon}{3}, \frac{p}{2n^{2k+2h(k,\varepsilon/3)}})$ 
9:       for all  $C \in \mathcal{C}_{L,\beta}$  with  $w^{L\beta}(e) \leq \eta_{k,\varepsilon/3} \cdot w^{L\beta}(C)$  for all  $e \in C$  do
10:         $P \leftarrow \text{DECOMPOSE}(C, w^{L\beta}, \varepsilon/3)$ 
11:        remove edges of weight 0 from  $P$ 
12:        add edges to  $K \cup P$  to obtain a Hamiltonian cycle  $H$ 
13:        add  $H$  to  $\mathcal{P}_{\text{TSP}}$ 
14:     for all  $i \leftarrow 1$  to  $k$  do
15:       remove the  $i$ th objective from  $w^L$  to obtain  $w'^L$ 
16:        $\mathcal{P}_{\text{TSP}}^{L,i} \leftarrow \text{MAXATSP-APPROX}(G, w'^L, k - 1, \frac{\varepsilon}{3}, \frac{p}{2n^{2k+2h(k,\varepsilon/3)}})$ 
17:       for all  $H' \in \mathcal{P}_{\text{TSP}}^{L,i}$  do
18:        remove edges of weight 0 from  $H'$ 
19:        add edges to  $H' \cup K$  to obtain a Hamiltonian cycle  $H$ 
20:        add  $H$  to  $\mathcal{P}_{\text{TSP}}$ 

```

Algorithm 2: Approximation algorithm for MC-Max-STSP.

Now we set $m = \frac{g(k,\varepsilon)}{k}$ and show that the set K contains enough weight after km iterations:

$$\begin{aligned}
1 - (1 - \eta_{k,\varepsilon} + \varepsilon^3)^m - \nu km &\geq \frac{2}{3} - \varepsilon &\Leftrightarrow (1 - \eta_{k,\varepsilon} + \varepsilon^3)^m + \nu km &\leq \frac{1}{3} + \varepsilon \\
&\Leftrightarrow (1 - \eta_{k,\varepsilon} + \varepsilon^3)^m &\leq \frac{1}{6} + \varepsilon \\
&\Leftrightarrow m &\geq \frac{\log(\frac{1}{6} + \varepsilon)}{\log(1 - \eta_{k,\varepsilon} + \varepsilon^3)}.
\end{aligned}$$

The implication holds since $\nu \leq \frac{1}{6 \cdot g(k,\varepsilon)}$ implies $\nu km \leq 1/6$. \square

We now have the key ingredients for our algorithm MAXSTSP-APPROX (Algorithm 2). What remains to be done is to specify ν more precisely and to analyze the approximation ratio as well as running-time and success probability of MAXSTSP-APPROX.

For ν , we assume that $0 < \nu \leq \frac{1}{6 \cdot g(k,\varepsilon)}$, which is necessary for Lemma 4.6. In addition, the set T of edges that we cannot force to be included in our Hamiltonian cycles must not be too heavy. Otherwise, we cannot maintain our approximation ratio. We know that $|T| \leq 2 \cdot g(k,\varepsilon)$ by Lemma 4.6. For $w(T) \leq \varepsilon \cdot w(\tilde{H})$, we need $w(e) \leq \frac{\varepsilon}{2 \cdot g(k,\varepsilon)} \cdot w(\tilde{H})$ for all $e \in T$. By Lemma 4.4, this can be achieved for $\ell \geq \lfloor \frac{2k \cdot g(k,\varepsilon)}{\varepsilon} \rfloor = h'(k,\varepsilon)$.

Let us summarize the above: In order to find the at most $g(k,\varepsilon)$ heavy edges, we need to examine subsets of edges of cardinality at most $g(k,\varepsilon)$. In order to keep T light, we have to consider at most $h'(k,\varepsilon) - 1$ edges to either side of these edges. Thus, we have to examine

subsets of edges of cardinality at most $(2h'(k, \varepsilon)) \cdot g(k, \varepsilon) = h(k, \varepsilon)$. This number depends only on k and ε , but not on the actual size of the input.

Now we are prepared to prove the main theorem of Section 4.2.

Theorem 4.7. *For every $k \geq 1$, $\varepsilon > 0$, MAXSTSP-APPROX is a randomized $\frac{2}{3} - \varepsilon$ approximation for k -criteria Max-STSP whose running-time for a success probability of at least $1 - p$ is polynomial in the input size and $\log(1/p)$.*

Proof. We will first estimate the approximation ratio given that all randomized computations are successful. After that, we will prove bounds for the error probability and the running-time. All proofs are by induction on k . For $k = 1$, the theorem holds since there is a deterministic, polynomial-time 61/81 approximation for mono-criterion Max-STSP. In the following, we assume that the theorem is correct for $k - 1$, all fixed $\varepsilon > 0$, and all $p > 0$, and we prove it for k , fixed $\varepsilon > 0$, and all $p > 0$.

Let \tilde{H} be an arbitrary Hamiltonian cycle. According to Lemma 4.6 and the considerations above, there exists sets $K, T \subseteq \tilde{H}$ such that $|K| + |T| \leq h(k, \varepsilon/3)$, K is a path cover and consists of at most $g(k, \varepsilon/3)$ paths, and T is the set of edges that connect K to the rest of \tilde{H} . Let $L \subseteq V$ be the set of vertices incident to edges in K , and let w^L be obtained from w by setting all edges incident to L to 0 (this includes in particular all edges in K , which have both endpoints in L , and K , which have one endpoint in L).

One of the following two properties holds:

1. There exists an i with $w_i(K) \geq (\frac{2}{3} - \frac{\varepsilon}{3}) \cdot w_i(\tilde{H})$.
2. For all $e \in \tilde{H}$, we have $w^L(e) \leq (\eta_{k, \varepsilon/3} - (\frac{\varepsilon}{3})^3) \cdot w^L(\tilde{H})$.

In the first case, there exists an $H' \in \mathcal{P}_{\text{TSP}}^{L, i}$ with $w_j^L(H') \geq (\frac{2}{3} - \frac{\varepsilon}{3}) \cdot w_j^L(\tilde{H})$ for all $j \in [k] \setminus \{i\}$. We remove all edges from H' that have weight 0 with respect to w^L , which does not affect $w^L(H')$. We obtain a Hamiltonian cycle H from $K \cup H'$ (which is a collection of paths) by adding appropriate edges. Then we have

$$w_i(H) \geq w_i(K) \geq \left(\frac{2}{3} - \frac{\varepsilon}{3}\right) \cdot w_i(\tilde{H})$$

and, for $j \neq i$,

$$\begin{aligned} w_j(H) &\geq w_j(K) + w_j^L(H') \geq w_j(K) + \left(\frac{2}{3} - \frac{\varepsilon}{3}\right) \cdot w_j^L(\tilde{H}) \geq \left(\frac{2}{3} - \frac{\varepsilon}{3}\right) \cdot (w_j^L(\tilde{H}) + w_j(K)) \\ &= \left(\frac{2}{3} - \frac{\varepsilon}{3}\right) \cdot (w_j(\tilde{H}) - w_j(T)) \geq \left(\frac{2}{3} - \frac{\varepsilon}{3}\right) \cdot \left(1 - \frac{\varepsilon}{3}\right) \cdot w_j(\tilde{H}) \geq \left(\frac{2}{3} - \varepsilon\right) \cdot w_j(\tilde{H}). \end{aligned}$$

In the second case, let $\beta_i = \max\{w_i^L(e) \mid e \in \tilde{H}\} \leq (\eta_{k, \varepsilon/3} - (\varepsilon/3)^3) \cdot w^L(\tilde{H})$. Then, according to Lemma 4.5 (applied to w^L and \tilde{H}), $\mathcal{C}_{L, \beta}$ contains a cycle cover C with $w^{L\beta}(C) \geq (1 - \frac{\varepsilon}{3}) \cdot w^L(\tilde{H})$ and $w^{L\beta}(e) \leq \eta_{k, \varepsilon/3} \cdot w^{L\beta}(\tilde{H})$. This cycle cover C can then be decomposed into a collection P of paths with $w^L(P) \geq (\frac{2}{3} - \frac{2\varepsilon}{3}) \cdot w^L(\tilde{H})$. We can assume that P does not contain any edges of weight 0. Together with K and some appropriate edges, this yields a Hamiltonian cycle H with

$$\begin{aligned} w(H) &\geq w(K) + w^L(P) \geq w(K) + \left(\frac{2}{3} - \frac{2\varepsilon}{3}\right) \cdot w^L(\tilde{H}) \geq \left(\frac{2}{3} - \frac{2\varepsilon}{3}\right) \cdot (w(\tilde{H}) - w(T)) \\ &\geq \left(\frac{2}{3} - \frac{2\varepsilon}{3}\right) \cdot \left(1 - \frac{\varepsilon}{3}\right) \cdot w(\tilde{H}) \geq \left(\frac{2}{3} - \frac{8\varepsilon}{9}\right) \cdot w(\tilde{H}) \geq \left(\frac{2}{3} - \varepsilon\right) \cdot w(\tilde{H}). \end{aligned}$$

What remains to be analyzed is the success probability and the running-time. By the induction hypothesis, the running-time of MAXSTSP-APPROX is polynomial in the input size and $\log(1/p)$ for a success probability of at least $1 - p$ for $k - 1$ objectives.

Lines 4 and 7 yield at most $n^{2k+2h(k,\varepsilon/3)}$ iterations. Then we compute that many approximate Pareto curves of cycle covers in line 8, each of polynomial cardinality. Some of these cycle covers are then decomposed to yield Hamiltonian cycles for \mathcal{P}_{TSP} . This can be implemented to run in polynomial time.

Lines 4 plus 14 yield at most $kn^{2h(k,\varepsilon/3)} \leq n^{2k+2h(k,\varepsilon/3)}$ iterations. Each iteration requires a call of MAXATSP-APPROX with $k - 1$ criteria, which is also polynomial.

We have at most $2n^{2k+2h(k,\varepsilon/3)}$ calls of a randomized procedure, each called with an error bound of $\frac{p}{2n^{2k+2h(k,\varepsilon/3)}}$. By a union bound, this gives an overall error probability of at most p . \square

5 Deterministic Approximations for 2-C-Max-STSP

The algorithms presented in the previous section are randomized due to the computation of approximate Pareto curves of cycles covers. So are all approximation algorithms for MC-Min-TSP that we are aware of with the exception of a simple $(2 + \varepsilon)$ approximation for MC-Min-STSP [25].

As a first step towards deterministic approximation algorithms for MC-Max-TSP, we present a deterministic $61/243 \approx 0.251$ approximation for 2-C-Max-STSP. The key insight for the results of this section is the following lemma.

Lemma 5.1. *Let M be a matching, let H be a collection of paths or a Hamiltonian cycle, and let w be edge weights. Then there exists a subset $P \subseteq H$ such that*

- (i) $P \cup M$ is a collection of paths or a Hamiltonian cycle (we call P in this case an M -feasible set) and
- (ii) $w(P) \geq w(H)/3$.

Proof. We prove the lemma by induction on $|M|$. For $|M| = 0$, we can choose $P = H$ and the lemma follows.

Now let $|M| = \ell > 0$ and assume that the lemma holds for all smaller sets M and all H . We prove that it also holds for $|M| = \ell$ by induction on $|H|$. The base case is $H = \emptyset$, for which the lemma obviously holds by setting $P = H$. Now assume that the lemma holds for $|M| = \ell$ and $|H| < m$ and that we have $|H| = m$.

We distinguish two cases. The first case is that $M \cap H \neq \emptyset$. Then we set $\tilde{P} = M \cap H$ and $H' = H \setminus M$. By the induction hypothesis, there exists a $P' \subseteq H'$ such that $w(P') \geq w(H')/3$ and P' is an M -feasible set. Since $\tilde{P} \subseteq M$, we have $P' \cup M = (P' \cup \tilde{P}) \cup M$. Thus, also $P = P' \cup \tilde{P}$ is an M -feasible set. Observing that $w(P) = w(P') + w(\tilde{P}) \geq w(H \cap M) + w(H \setminus M)/3 \geq w(H)/3$ completes the first case.

The second case is that $M \cap H = \emptyset$. Let $e = \operatorname{argmax}\{w(e) \mid e \in H\}$ be a heaviest edge of H , and let $f_1, f_2 \in H$ be the two edges of H that are incident to e . Let $H' = H \setminus \{e, f_1, f_2\}$. (It can happen that f_1 or f_2 do not exist, namely if H is not a Hamiltonian cycle but a collection of paths. But this is fine.)

Let us first treat the case that e is incident to two edges $z_1, z_2 \in M$ of the matching. (The matching M is not necessarily perfect.) Then we contract z_1 and z_2 to a single edge z that connects the two endpoints of z_1 and z_2 that are not incident to e and remove the two vertices

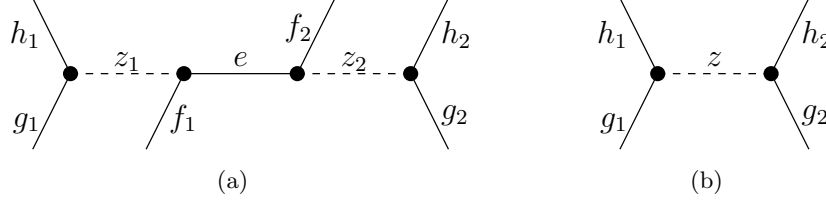


Figure 2: Contraction for the proof of Lemma 5.1: We keep e and remove f_1, f_2 . Then we can contract z_1 and z_2 to z .

incident to e (see Figure 2). Let $M' = (M \setminus \{z_1, z_2\}) \cup \{z\}$. Since e, f_1, f_2 are removed, H' and M' are a valid instance for the lemma, i.e., M' is a matching and H' is a collection of paths (H' cannot be a Hamiltonian cycle). We can apply the induction hypothesis since $|M'| < |M|$.

In this way, we obtain an M' -feasible set $P' \subseteq H'$ with $w(P') \geq w(H')/3$. Set $P = P' \cup \{e\}$. Since $w(e) \geq w(\{e, f_1, f_2\})/3$, we have $w(P) \geq w(H)/3$. It remains to be proved that P is M -feasible: Every vertex in $M \cup P$ has a degree of at most 2 by construction. Thus, the only possibility for P not to be M -feasible is that it contains a non-Hamiltonian cycle. By the induction hypothesis, $M' \cup P'$ does not contain such a cycle. Thus, also $M \cup P$ does not contain a cycle, since $M \cup P$ is obtained from $M' \cup P'$ just by replacing z by z_1, e , and z_2 .

What remains to be considered is the case the e is not incident to two edges $z_1, z_2 \in M$. Then we consider the shortest path in $e_1, \dots, e_q \in H$ of edges in H that includes e such that e_1 and e_q are incident to any edges $z_1, z_2 \in M$. The reasoning above holds in exactly the same way if replace e by the path e_1, \dots, e_q , and we put e_1, \dots, e_q into P . If no such path exists, then either $M = \emptyset$, which we have already dealt with, or the path containing e ends somewhere at a vertex of degree 1 in $H \cup M$. In the latter case, we can simply put the whole path into P . \square

Lemma 5.1 yields tight bounds for the existence of approximate Pareto curves with only a single element. This is the purpose of the following theorem.

Theorem 5.2. (a) *For every undirected complete graph G with edge weights w_1 and w_2 , there exists a Hamiltonian cycle H such that $\{H\}$ is a $1/3$ approximate Pareto curve for 2-C-Max-STSP.*

(b) *Part (a) is tight: There exists a graph G with edge weights w_1 and w_2 such that, for all $\varepsilon > 0$, no single Hamiltonian tour of G is a $(1/3 + \varepsilon)$ approximate Pareto curve.*

Before embarking on the proof of the theorem, let us make some remarks. The question how well a single element can approximate a whole Pareto curve has also been addressed by Ehrgott [11]: He proved that for MC-Min-STSP, there is always a single Hamiltonian cycle H such that the norm of $w(H)$ is at most twice as large as the norm of $w(\tilde{H})$ for any Hamiltonian cycle \tilde{H} , i.e., $\|w(H)\| \leq 2 \cdot \|w(\tilde{H})\|$. This, however, does not imply that $\{H\}$ is already an approximate Pareto curve. Instead, single-element approximate Pareto curves exist for no other variant of multi-criteria TSP than 2-C-Max-STSP: For k -C-Max-STSP for $k \geq 3$, we can consider a vertex incident to three edges of weight $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$, respectively. All other edges of the graph have weight 0. Then no single Hamiltonian cycle can have positive weight with respect to all three objectives simultaneously. Similarly, no such result is possible for k -C-Max-ATSP and for k -C-Min-TSP for any $k \geq 2$.

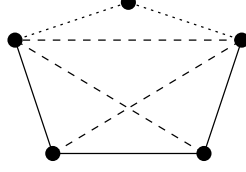


Figure 3: The graph for Theorem 5.2(b). Solid edges are of weight $(1, 0)$, dashed edges are of weight $(0, 1)$, all other edges (dotted or not drawn) have weight $(0, 0)$.

Proof. Let H_1 and H_2 Hamiltonian cycles of G such that H_1 maximizes w_1 and H_2 maximizes w_2 . Then there exists a matching $M \subseteq H_1$ with $w(M) \geq w(H_1)/3$. (We can actually get $w(H_1)/2$ if G has an even number of vertices and $\frac{n-1}{2n} \cdot w(H_1)$ if the number n of G 's vertices is odd. This, however, does not improve the result.) We apply Lemma 5.1 with $H = H_2$ and obtain an M -feasible set $P \subseteq H_2$. From M and P , we obtain a Hamiltonian cycle $H' \supseteq M \cup P$: Either $M \cup P$ is already a Hamiltonian cycle, then nothing has to be done. Or $M \cup P$ is a collection of paths. Then we add appropriate edges to obtain H' . We claim that $\{H'\}$ is a $1/3$ approximate Pareto curve: Let \tilde{H} be any Hamiltonian tour. Then

$$w_1(H') \geq w_1(M) \geq w_1(H_1)/3 \geq w_1(\tilde{H})/3$$

and

$$w_2(H') \geq w_2(P) \geq w_2(H_2)/3 \geq w_2(\tilde{H})/3.$$

To finish the proof, let us show that the $1/3$ bound is tight, i.e., in general, no single solution can be better than a $1/3$ approximate Pareto curve. The proof is by contradiction. Consider the graph shown in Figure 3: The solid edges have weight $(1, 0)$, the dashed edges have weight $(0, 1)$. All other edges (two are dotted, the other edges are not shown) are of weight $(0, 0)$.

The solid edges plus the dotted edges form a Hamiltonian cycle of weight $(3, 0)$. The dashed edges plus the dotted edges form a Hamiltonian cycle of weight $(0, 3)$. Assume that there exists a Hamiltonian cycle H such that $\{H\}$ is a $1/3 + \varepsilon$ approximate Pareto curve for our instance. Then H must contain at least two solid edges (otherwise $w_1(H) \leq 1$) and at least two dashed edges (otherwise $w_2(H) \leq 1$). In order to include the top vertex, we also need two edges of weight $(0, 0)$. Thus, in total at least six edges are needed, but any Hamiltonian cycle on a five-vertex graph contains only five vertices – contradiction. \square

Lemma 5.1 and Theorem 5.2 are constructive in the sense that, given a Hamiltonian cycle H_2 that maximizes w_2 , the tour H can be computed in polynomial time. A matching M with $w_1(M) \geq w_1(H_1)/3$ can be computed in cubic time. However, since we cannot compute an optimal H_2 efficiently, the results cannot be exploited directly to get an algorithm. Instead, we use an approximation algorithm for finding a Hamiltonian tour with as much weight with respect to w_2 as possible. Using the $61/81$ approximation algorithm for Max-STSP [10], we obtain Algorithm 3 (which in particular is an algorithmic version of Lemma 5.1) and the following theorem.

Theorem 5.3. *BiMAXSTSP-APPROX is a deterministic $61/243$ approximation algorithm with running-time $O(n^3)$ for 2-C-Max-STSP.*

Proof. The running-time is dominated by the running-time of the $61/81$ approximation for Max-STSP by Chen et al. [10] and the time for computing the matching, both of which is $O(n^3)$. The approximation ratio follows from $\frac{61}{81} \cdot \frac{1}{3} = \frac{61}{243}$. \square

```

 $\mathcal{P}_{\text{TSP}} \leftarrow \text{BiMAXSTSP-APPROX}(G, w_1, w_2)$ 
input: undirected complete graph  $G = (V, E)$ , edge weights  $w_1, w_2 : E \rightarrow \mathbb{N}^k$ 
output: a 61/243 approximate Pareto curve  $H$ 
1: compute a maximum-weight matching  $M$  with respect to  $w_1$ 
2: compute a 61/81 approximate Hamiltonian cycle  $H_2$  with respect to  $w_2$ 
3:  $P \leftarrow H_2 \cap M$ 
4:  $M' \leftarrow M$ 
5:  $H_2 \leftarrow H_2 \setminus P$ 
6: while  $H_2 \neq \emptyset$  do
7:    $e \leftarrow \text{argmax}\{w_2(e') \mid e' \in H_2\}$ 
8:   extend  $e$  to a path  $e_1, \dots, e_q \in H_2$  such that only  $e_1$  and  $e_q$  are incident to edges
      $z_1, z_2 \in M'$  or the path cannot be extended anymore
9:    $P \leftarrow P \cup \{e_1, \dots, e_q\}$ 
10:   $H_2 \leftarrow H_2 \setminus \{e_1, \dots, e_q\}$ 
11:  if  $z_1$  or  $z_2$  exists then
12:    let  $f_1, f_2 \in H_2$  be the two edges extending the path if they exist
13:     $H_2 \leftarrow H_2 \setminus \{f_1, f_2\}$ 
14:    if both  $z_1$  and  $z_2$  exist then
15:      contract  $z_1$  and  $z_2$  to  $z$ 
16:       $M' \leftarrow (M' \setminus \{z_1, z_2\}) \cup \{z\}$ 
17: let  $H$  be a Hamiltonian cycle obtained from  $P \cup M$ 

```

Algorithm 3: Approximation algorithm for 2-C-Max-STSP.

For the special case of metric 2-C-Max-STSP, i.e., the edge weights are restricted to fulfil the triangle inequality, we obtain the slightly better approximation ratio of $7/24 > 0.29$ if we replace the 61/81 approximation with the $7/8$ approximation for metric Max-STSP by Kowalik and Mucha [24].

Finally, we note that the bound of $1/3$ of Theorem 5.2 holds only for small instances. If the weight of the optimum Hamiltonian cycles is distributed among many edges, then $(\frac{3}{8} - \delta)$ approximate Pareto curves exist for some $\delta > 0$. The following lemma states this more precisely.

Lemma 5.4. *Let $G = (V, E)$ be an undirected complete graph, and let w_1 and w_2 be edge weights. Let H_1 and H_2 be two Hamiltonian cycles maximizing w_1 and w_2 , respectively. Let $c = \max\{\frac{w_i(e)}{w_i(H_i)} \mid i \in \{1, 2\}, e \in H_i\}$ be the maximum contribution of a single edge to $w_1(H_1)$ or $w_2(H_2)$.*

Then there exists a Hamiltonian cycle H that is a single-element $(\frac{3}{8} - \delta)$ approximate Pareto curve for $\delta > \frac{3c}{8} + \sqrt{\frac{c}{4} \cdot \ln 2}$.

Proof. We prove that there exists a collection P of paths such that $w_i(P) \geq (\frac{3}{8} - \delta) \cdot w_i(H_i)$ for $i = 1, 2$. There exists matchings $M_i \subseteq H_i$ with $w_i(M_i) \geq \frac{w_i(H_i) - c}{2}$ for $i \in \{1, 2\}$. Consider the graph $(V, M_1 \cup M_2)$. Any component of this graph is either a single edge that is common to M_1 and M_2 . Then we put this edge into P . Or it is a cycle of even length, consisting in turn of edges of M_1 and M_2 . Then we have to remove an edge to obtain a path. Let c_1, \dots, c_ℓ be these cycles.

For every cycle, we choose randomly to remove either the lightest edge from M_1 or the lightest edge of M_2 . Thus, the contribution of c_j to w_i is a random variable X_j that assumes values in an interval of at most c . We have $\mathbb{E}(\sum_{j=1}^{\ell} X_j) \geq \frac{3}{4} \cdot w_i(M_i) \geq \frac{3}{8} w_i(H_i) - \frac{3c}{8}$. Thus,

by Hoeffding's inequality, the probability that $\mathbb{E}(\sum_{j=1}^{\ell} X_j) < (\frac{3}{8} - \delta) \cdot w_i(H_i)$ is at most

$$\exp\left(-\frac{2 \cdot (\delta - \frac{3c}{8})^2}{\sum_{i=1}^{\ell} (b_j - a_j)^2}\right),$$

where a_j is $w_i(c_j)$ minus the weight of the lighter edge of M_i with respect to w_i and $b_j = w_i(c_j)$. We have $b_j - a_j \leq c$ and $\sum_{j=1}^{\ell} b_j = 1$ as well as $b_j \leq 2a_j$, which yields

$$\frac{1}{2} \geq \sum_{j=1}^{\ell} b_j - a_j \geq \ell c.$$

Thus,

$$\frac{c}{2} \geq \sum_{j=1}^{\ell} (b_j - a_j)^2.$$

Altogether,

$$\begin{aligned} \exp\left(-\frac{2 \cdot (\delta - \frac{3c}{8})^2}{\sum_{i=1}^{\ell} (b_j - a_j)^2}\right) < \frac{1}{2} &\Leftarrow \exp\left(-\frac{4 \cdot (\delta - \frac{3c}{8})^2}{c}\right) < \frac{1}{2} \\ &\Leftrightarrow \delta > \frac{3c}{8} + \sqrt{\frac{c}{4} \cdot \ln 2}. \end{aligned}$$

Thus, by a union bound, the probability that $w_1(P) < \frac{3}{8} - \delta$ or $w_2(P) < \frac{3}{8} - \delta$ is smaller than 1, which implies the existence of such a P . \square

For instance, if $c \leq 0.01$, i.e., no edge contains more than 1% of the weight of an optimal Hamiltonian cycle, then a single-element 0.37 approximate Pareto curve exists. If $c \leq 0.18$, then a single-element 0.35 approximate Pareto curve exists.

6 Approximation Algorithm for MC-Min-ATSP

Now we turn to MC-Min-ATSP and MC-Min- γ -ATSP, i.e., Hamiltonian cycles of *minimum* weight are sought in directed graphs. Algorithm 4 is an adaptation of the algorithm of Frieze et al. [16] to multi-criteria ATSP. Therefore, we briefly describe their algorithm: We compute a cycle cover of minimum weight. If this cycle cover is already a Hamiltonian cycle, then we are done. Otherwise, we choose an arbitrary vertex from every cycle. Then we proceed recursively on the subset of vertices thus chosen to obtain a Hamiltonian cycle that contains all these vertices. The cycle cover plus this Hamiltonian cycle form an Eulerian graph. We traverse the Eulerian cycle and take shortcuts whenever visiting vertices more than once. The approximation ratio achieved by this algorithm is $\log_2 n$ for Min-ATSP [16] and $1/(1 - \gamma)$ for Min- γ -ATSP [7].

To approximate MC-Min-ATSP, we use MINATSP-APPROX (Algorithm 4), which proceeds as follows: We compute an approximate Pareto curve of cycle covers. This is done by MINCC-APPROX, where MINCC-APPROX(G, w, k, ε, p) computes a $(1 + \varepsilon)$ approximate Pareto curve of cycle covers of G with weights w with a success probability of at least $1 - p$ in time polynomial in the input size, $1/\varepsilon$, and $\log(1/p)$. (Of course, the aim is now to find

```

 $\mathcal{P}_{\text{TSP}} \leftarrow \text{MINATSP-APPROX}(G, w, k, \varepsilon)$ 
input: directed complete graph  $G = (V, E)$  with  $n = |V|$ ,  $k \geq 1$ , edge weights  $w : E \rightarrow \mathbb{N}^k$ ,  $\varepsilon > 0$ 
output:  $(\log n + \varepsilon)$  approximate Pareto curve for  $k$ -C-Min-ATSP or  $(\frac{1}{1-\gamma} + \varepsilon)$  approximate Pareto curve for  $k$ -C-Min- $\gamma$ -ATSP with a probability of at least  $1/2$ 
1:  $\varepsilon' \leftarrow \varepsilon^2 / \log^3 n$ ;  $\mathcal{F} \leftarrow \emptyset$ ;  $j \leftarrow 1$ 
2:  $\mathcal{C} \leftarrow \text{MINCC-APPROX}(G, w, k, \varepsilon', \frac{1}{2Q \log n})$   $\triangleright Q$  is defined in Lemma 6.2
3:  $\mathcal{P}_0 \leftarrow \{(C, w(C), V, \perp) \mid C \in \mathcal{C}\}$ 
4: while  $\mathcal{P}_{j-1} \neq \emptyset$  do
5:    $\mathcal{P}_j \leftarrow \emptyset$ 
6:   for all  $\pi = (C', w', V', \pi') \in \mathcal{P}_{j-1}$  do
7:     if  $(V', C')$  is connected then
8:        $\mathcal{F} \leftarrow \mathcal{F} \cup \{(C', w', V', \pi')\}$ 
9:     else
10:      select one vertex of every component of  $(V', C')$  to obtain  $\tilde{V}$ 
11:       $\tilde{\mathcal{C}} \leftarrow \text{MINCC-APPROX}(G, w, k, \varepsilon', \frac{1}{2Q \log n})$ 
12:       $\mathcal{P}_j \leftarrow \mathcal{P}_j \cup \{(\tilde{C}, \tilde{w}, \tilde{V}, \pi) \mid \tilde{C} \in \tilde{\mathcal{C}}, \tilde{w} = w' + \gamma^j \cdot w(\tilde{C})\}$ 
13:   while there are  $\pi', \pi'' \in \mathcal{P}_j$  with the same  $\varepsilon'$ -signature do
14:     remove one of them arbitrarily
15:    $j \leftarrow j + 1$ 
16:  $\mathcal{P}_{\text{TSP}} \leftarrow \emptyset$ 
17: for all  $(C', w', V', \pi') \in \mathcal{F}$  do
18:    $H \leftarrow C'$ 
19:   while  $\pi' = (C'', w'', V'', \pi'') \neq \perp$  do
20:     construct a Hamiltonian cycle  $H'$  on  $V''$  from  $H \cup C''$  by taking shortcuts such that all edges of  $H$  are removed
21:      $\pi' \leftarrow \pi''$ 
22:      $H \leftarrow H'$ 
23:  $\mathcal{P}_{\text{TSP}} \leftarrow \mathcal{P}_{\text{TSP}} \cup \{H\}$ 

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Algorithm 4: Approximation algorithm for MC-Min-ATSP and MC-Min- γ -ATSP.

cycle covers of minimum weight.) Then we iterate by computing approximate Pareto curves of cycle covers on vertex sets V' for every cycle cover C in the previous set. The set V' contains exactly one vertex of every cycle of C . Unfortunately, it can happen that we construct a super-polynomial number of solutions in this way. To cope with this, we remove some intermediate solutions if there are other intermediate solutions whose weight is close by. We call this process *sparsification*. It is performed in lines 13 and 14 of Algorithm 4 and based on the following observation: Let $\varepsilon > 0$, and consider H of weight $w(H) \in \mathbb{N}^k$. For every $i \in \{1, \dots, k\}$, there is a unique $\ell_i \in \mathbb{N}$ such that $w_i(H) \in [(1 + \varepsilon)^{\ell_i}, (1 + \varepsilon)^{\ell_i + 1})$. We call the vector $\ell = (\ell_1, \dots, \ell_k)$ the ε -signature of H and of $w(H)$. Since $w(H) \leq 2^{p(N)}$, where N is the size of the instance, ℓ_i is bounded by a polynomial $q(N, 1/\varepsilon)$. There are at most q^k different ε -signatures, which is polynomial for fixed k . To get an approximate Pareto curve, we can restrict ourselves to have at most one solution with any specific ε -signature.

In lines 2 and 3 of Algorithm 4, an initial $(1 + \varepsilon')$ approximate Pareto curve of cycle covers is computed. In the loop in lines 4 to 15, the algorithm computes iteratively Pareto curves of cycle covers. The set \mathcal{P}_j contains *configurations* $\pi = (C', w', V', \pi')$, where C' is a cycle

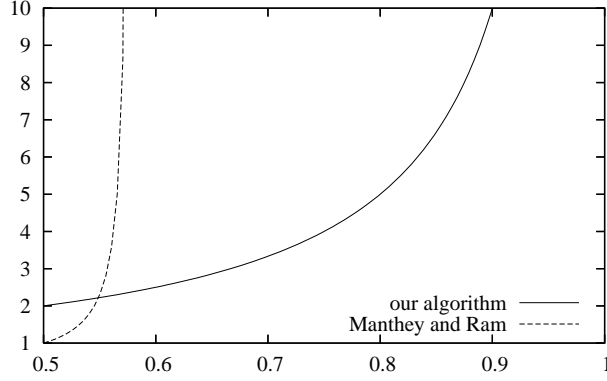


Figure 4: Comparison of the approximation ratio subject to γ for MC-Min- γ -ATSP of Manthey and Ram’s algorithm [25] and our algorithm.

cover on V' , π' is the predecessor configuration, and w' is the weight of C' plus the weight of its predecessor cycle covers, each weighted with an appropriate power of γ . (We define the ε' -signature of $\pi = (C', w', V', \pi')$ to be the ε' -signature of w' .) The reason for these weights will become clear in the analysis of the approximation ratio of the algorithm. If, in the course of this computation, we obtain Hamiltonian cycles, these are put into \mathcal{F} (line 8). In lines 13 and 14, the sparsification takes place. Finally, in lines 16 to 23, Hamiltonian cycles are constructed from the cycle covers computed.

Let us now come to the analysis of the algorithm. Our goal is to prove the following result, which follows from Lemmas 6.2, 6.3, and 6.6 below. MINATSP-APPROX is the first approximation algorithm for MC-Min-ATSP and for MC-Min- γ -ATSP for $\gamma \geq \sqrt{1/3} \approx 0.58$. Furthermore, for $\gamma > 0.55$, it improves over the previously known algorithm, which works only for $1/2 \leq \gamma < \sqrt{1/3} \approx 0.58$ (see Figure 4 for a comparison of the approximation ratios).

Theorem 6.1. *For every $\varepsilon > 0$, Algorithm 4 is a randomized $(\log n + \varepsilon)$ approximation for MC-Min-ATSP and a randomized $(\frac{1}{1-\gamma} + \varepsilon)$ approximation for MC-Min- γ -ATSP for $\gamma \in [\frac{1}{2}, 1)$ with a success probability of at least $1/2$. Its running-time is polynomial in the input size and $1/\varepsilon$.*

We observe that for every j and every $\pi = (C', w', V', \pi') \in \mathcal{P}_j$, we have $|V'| \leq n/2^j$. For $j = 0$, this holds by construction (line 3). For $j > 0$ and $\pi' = (C'', V'', w'', \pi'') \in \mathcal{P}_{j-1}$, we have $|V''| \leq n/2^{j-1}$ by the induction hypothesis. Since every cycle involves at least two vertices, we have $|V'| \leq |V''|/2 \leq n/2^j$. This yields also that \mathcal{P}_j is empty for $j \geq \lfloor \log n \rfloor$: Assume to the contrary that such a \mathcal{P}_j contained a configuration $(\tilde{C}, \tilde{V}, \tilde{w}, \pi)$. Then $|\tilde{V}| \leq n/2^{\lfloor \log n \rfloor} < 2$. Thus, $|\tilde{V}| \leq 1$. Let $\pi = (C', w', V', \pi')$, then this implies that (V', C') had been connected. Hence, we would enter line 8 rather than lines 10 to 12 – a contradiction.

Let us now analyze the running-time. After that, we examine the approximation performance and finally the success probability.

Lemma 6.2. *The running-time of Algorithm 4 is polynomial in the input size and $1/\varepsilon$.*

Proof. Let N be the size of the instance at hand, and let $Q = Q(N, 1/\varepsilon')$ be a two-variable polynomial that bounds the number of different ε' -signatures of solutions for instances of size

at most N . We abbreviate “polynomial in the input size and $1/\varepsilon$ ” simply by “polynomial.” This is equivalent to “polynomial in the input size and $1/\varepsilon'$ ” by the choice of ε' .

The approximate Pareto curves can be computed in polynomial time with a success probability of at least $1 - (2Q \log n)^{-1}$ by executing the randomized FPTAS $\lceil \log(2Q \log n) \rceil$ times. Thus, all operations can be implemented to run in polynomial time provided that the cardinalities of all sets \mathcal{P}_j are bounded from above by a polynomial Q for all j . Then, for each j , at most Q approximate Pareto curves of cycle covers are constructed in line 11, each one in polynomial time.

For every ε' -signature and every j , the set \mathcal{P}_j contains at most one cycle cover with that specific ε' -signature. The lemma follows since the number Q of different ε' -signatures is bounded by a polynomial. \square

Let us now analyze the approximation ratio. First, we will assume that all randomized computations of $(1 + \varepsilon')$ approximate cycle covers are successful. After that we analyze the probability that one of them fails.

Lemma 6.3. *Assume that in all executions of line 11 of Algorithm 4 an $(1 + \varepsilon')$ approximate Pareto curve of cycle covers is successfully computed. Then Algorithm 4 achieves an approximation ratio of $\sum_{j=0}^{\lfloor \log n \rfloor - 1} \gamma^j + \varepsilon$ for MC-Min- γ -ATSP for $\gamma \in [\frac{1}{2}, 1]$.*

If $\gamma < 1$, then $\sum_{j=0}^{\lfloor \log n \rfloor - 1} \gamma^j \leq \sum_{j=0}^{\infty} \gamma^j = \frac{1}{1-\gamma}$. If $\gamma = 1$, then $\sum_{j=0}^{\lfloor \log n \rfloor - 1} \gamma^j = \lfloor \log n \rfloor \leq \log n$. Thus, we obtain the approximation ratios claimed in Theorem 6.1.

Proof. Let $r = \sum_{j=0}^{\lfloor \log n \rfloor - 1} \gamma^j$, and let \tilde{H} be any Hamiltonian cycle on V . We have to show that the set \mathcal{P}_{TSP} of solutions computed by Algorithm 4 contains a Hamiltonian cycle H with $w(H) \leq (r + \varepsilon) \cdot w(\tilde{H})$.

Lemma 6.4. *Let $\pi = (C', w', V', \pi') \in \mathcal{F}$. Then, in lines 16 to 23, a Hamiltonian cycle H is constructed from π with $w(H) \leq w'$.*

Proof. Assume that $\pi_\ell = \pi$ for some ℓ and that $\pi_j = (C_j, V_j, w^{(j)}, \pi_{j-1})$ for $j \in \{0, \dots, \ell\}$. (We have $\pi_{-1} = \perp$, $C' = C_\ell$, $V' = V_\ell$, $w^{(\ell)} = w'$, $\pi_{\ell-1} = \pi'$, and $V_0 = V$.) Let $H_\ell = C_\ell$ be the Hamiltonian cycle on V_ℓ . In lines 16 to 23, we iteratively construct Hamiltonian cycles H_j on V_j from C_j and H_{j+1} . We have $w(H_j) \leq w(C_j) + \gamma \cdot w(H_{j+1})$ by the γ -triangle inequality. Thus, $w(H) = w(H_0) \leq \sum_{j=0}^{\ell} \gamma^j \cdot w(C_j) = w'$ by the definition of w' . \square

What remains to be proved is that, for every Hamiltonian cycle \tilde{H} , there exists a $\pi = (C', w', V', \pi')$ in \mathcal{F} such that $w' \leq (r + \varepsilon) \cdot w(\tilde{H})$.

Lemma 6.5. *For every ℓ , there exists a configuration $\pi = (C', w', V', \pi') \in \mathcal{P}_\ell \cup \mathcal{F}$ with $w' \leq (1 + \varepsilon')^{\ell+1} \cdot \sum_{j=0}^{\ell} \gamma^j \cdot w(\tilde{H})$.*

Proof. The proof is by induction on ℓ . For $\ell = 0$, the lemma boils down to the existence of a configuration $(C', w', V', \perp) \in \mathcal{P}_0$ with $w(C') \leq (1 + \varepsilon') \cdot w(\tilde{H})$. Such a C' exists because in line 2, a $(1 + \varepsilon')$ approximate Pareto curve of cycle covers is computed.

Now assume that the lemma holds for $\ell - 1$. If \mathcal{F} contains a configuration (C', w', V', π') that satisfies the lemma for $\ell - 1$, then we are done since $(1 + \varepsilon')^\ell \cdot \sum_{j=0}^{\ell-1} \gamma^j \leq (1 + \varepsilon')^{\ell+1} \cdot \sum_{j=0}^{\ell} \gamma^j$. Otherwise, $\mathcal{P}_{\ell-1}$ contains a configuration $\pi' = (C'', V'', w'', \pi'')$ with $w'' \leq (1 + \varepsilon')^\ell \cdot \sum_{j=0}^{\ell-1} \gamma^j \cdot w(\tilde{H})$. Let V' be the set of vertices constructed from π' in line 10, and let \tilde{H}' be \tilde{H} restricted

to V' by taking shortcuts. By the triangle inequality, we have $w(\tilde{H}') \leq w(\tilde{H})$. After line 11, \mathcal{C} contains a cycle cover C' with $w(C') \leq (1 + \varepsilon') \cdot w(\tilde{H}')$. Let $\pi = (C', w', V', \pi')$ with $w' = w'' + \gamma^\ell \cdot w(C')$. Then

$$w' \leq \left((1 + \varepsilon')^\ell \cdot \sum_{j=0}^{\ell-1} \gamma^j + \gamma^\ell \cdot (1 + \varepsilon') \right) \cdot w(\tilde{H}).$$

What remains to be analyzed is the sparsification in lines 13 to 14. If $\pi \in \mathcal{P}_j$ after sparsification, we are done since

$$(1 + \varepsilon')^\ell \cdot \sum_{j=0}^{\ell-1} \gamma^j + (1 + \varepsilon') \cdot \gamma^\ell \leq (1 + \varepsilon')^{\ell+1} \cdot \sum_{j=0}^{\ell} \gamma^j.$$

Otherwise, \mathcal{P}_ℓ contains a $\tilde{\pi} = (\tilde{C}', \tilde{w}', \tilde{V}', \tilde{\pi}')$ with the same ε' -signature as π . Thus,

$$\tilde{w}' \leq (1 + \varepsilon') \cdot \left((1 + \varepsilon')^\ell \cdot \sum_{j=0}^{\ell-1} \gamma^j + \gamma^\ell \cdot (1 + \varepsilon') \right) \cdot w(\tilde{H}) \leq (1 + \varepsilon')^{\ell+1} \cdot \sum_{j=0}^{\ell} \gamma^j \cdot w(\tilde{H}),$$

and $\tilde{\pi}$ fulfills the requirements of the lemma. \square

Let r be defined as in the beginning of the proof of Lemma 6.3. By Lemmas 6.4 and 6.5, we obtain an approximation ratio of $(1 + \varepsilon')^{\lfloor \log n \rfloor} \cdot r$ since \mathcal{P}_ℓ is empty for $\ell \geq \lfloor \log n \rfloor$. What remains to be proved is that this is at most $r + \varepsilon$:

$$r \cdot (1 + \varepsilon')^{\lfloor \log n \rfloor} \leq r \cdot \left(1 + \frac{\varepsilon^2}{\log^3 n} \right)^{\log n} \leq r \cdot \exp \left(\frac{\varepsilon^2}{\log^2 n} \right) \leq r \cdot \left(1 + \frac{\varepsilon}{\log n} \right) \leq r + \varepsilon.$$

The first inequality follows from our choice of ε' . The second inequality holds since $(1 + \frac{x}{y})^y \leq \exp(x)$. The third inequality holds because $\exp(x^2) \leq 1 + x$ for $x \in [0, 0.7]$ (we assume $\varepsilon/\log n < 0.7$ without loss of generality.) The fourth inequality holds since $r \leq \log n$. \square

Lemma 6.6. *The probability that, in a run of MINATSP-APPROX (Algorithm 4), in the execution of every line 2 and in every execution of line 11 an ε' approximate Pareto curve of cycle covers is successfully computed is at least $1/2$.*

Proof. Lines 11 and 2 of Algorithm 4 is executed at most $Q \cdot \log n$ times, where Q is an upper bound for the number of different ε' -signatures of solutions of instances of size at most N . (Q is polynomial in N and $1/\varepsilon$.) Each execution fails with a probability of at most $\frac{1}{2Q \log n}$. Thus by a union bound, the probability that one of them fails is at most $\frac{Q \log n}{2Q \log n} = 1/2$. \square

MINATSP-APPROX uses randomization only for MINCC-APPROX. Thus, Lemma 6.6 immediately yields that our algorithm has a success probability of at least $1/2$ in compliance with Theorem 6.1 and the definition of a randomized approximation algorithm. The success probability can be amplified to $1 - p$ for arbitrarily small $p > 0$ by running the algorithm $O(\log(1/p))$ times.

7 Conclusions

We have presented approximation algorithms for almost all variants of multi-criteria TSP. The approximation ratios of our algorithms are independent of the number k of criteria and come close to the currently best ratios for TSP with a single objective. Furthermore, all of our algorithms work for any number of criteria.

In particular, we have presented a factor $\frac{1}{2} - \varepsilon$ approximation algorithm for MC-Max-ATSP and a factor $\frac{2}{3} - \varepsilon$ approximation algorithm for MC-Max-STSP. The algorithms are randomized, and their running-time is polynomial in the input size for all fixed $\varepsilon > 0$ and k . In terms of approximation ratios, this is close to the best known ratios for mono-criterion TSP ($\frac{2}{3}$ for Max-ATSP and $\frac{61}{81}$ for Max-STSP). Our algorithms improve upon the previous algorithms for these problems that achieve ratios of $1/(k+1) + \varepsilon$ and $1/k + \varepsilon$, and they give an affirmative answer to the question raised by Bläser et al. [6] whether there exist algorithms with approximation ratio $\Omega(1/\log k)$.

Furthermore, we have presented a randomized $\log_2 n + \varepsilon$ approximation for MC-Min-ATSP. The approximation ratio is asymptotically equal to the best known ratio of $\frac{2}{3} \cdot \log_2 n$ for Min-ATSP. Finally, we devised a deterministic $61/243$ approximation for 2-C-Min-STSP with cubic running-time, and we proved that for 2-C-Min-STSP, there always exists a $1/3$ approximate Pareto curve that consists of a single element.

Most approximation algorithms for multi-criteria TSP use randomness since computing approximate Pareto curves of cycle covers requires randomness. This raises the question of whether there are algorithms (particularly designed) for multi-criteria TSP that are faster, deterministic, and achieve better approximation ratios.

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